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Physics of Stars (1): Stellar Structure

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1 Equations of Stellar Structure

The task of stellar astrophysics is to explain the systematic properties that we have discussed in the 'Observational Astronomy' section. A star is characterised by its mass M, radius R, luminosity L, and temperature T. We'll also see that its composition or *metallicity* Z also matters ('metals' are anything other than H & He: the Sun has about 2% of its mass in metals and 25% in He). Some of these quantities are related. For instance, since a star shines like a black body:

$$L = 4\pi R^2\,\sigma\,T^4_{\mathrm{surface}}, \tag{1}$$

where $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}K^{-4}}$ is the *Stefan-Boltzmann constant*. The mass and radius are related through the average density $\bar{\rho}$:

$$M = 4\pi R^3\bar{\rho}/3. \tag{2}$$

More generally, if $\rho(r)$ denotes the local density of the star at a radius r from its centre, then the mass ΔM of the star in a shell of thickness Δr is $\Delta M = 4\pi r^2\rho(r)\Delta r$, or

$$\boxed{\frac{dM(r)}{dr} = 4\pi r^2\rho(r)} \tag{3}$$

This is the **first equation of stellar structure** (mass conservation, or the **equation of continuity**). $M(r)$ means the mass interior to radius r.

Now, the local density $\rho(r)$ and temperature $T(r)$ are also related. For a star on the main sequence, its hot interior behaves like an *ideal gas*:

$$P(r) = n(r)\,kT(r) \quad \Rightarrow \quad P(r) = [\rho(r)/\bar{m}]\,kT(r), \tag{4}$$

where $n(r)$ is the number density of all particles (electrons plus nuclei, since the material in the Sun is ionized). The mass $\bar{m}$ is the *mean mass per particle*: $n(r) = \rho(r)/\bar{m}$. We shall generally assume $\bar{m} = m_p/2$: most of the Sun is Hydrogen plasma, and each electron-proton pair weighs basically just the proton mass – or $m_p/2$ on average.

The internal pressure of the star varies with radius, and this pressure gradient supports the star against its own gravity. Consider a parcel of gas of mass m inside the star, with area A and radial extent ΔR: The pressure force acting on the bottom is $F_{\rm bottom} = P(r)A$; on the top, it is

$$F_{\rm top} = P(r + \Delta r)A \simeq P(r)A + \frac{dP}{dr}\Delta r\, A. \tag{5}$$

The net pressure force acting is thus

$$F_{\rm bottom} - F_{\rm top} = P(r)A - \left[P(r)A + \frac{dP}{dr}\Delta r\, A\right] = -\frac{dP}{dr}\Delta r\, A. \tag{6}$$

In **hydrostatic equilibrium**, this net pressure force must balance gravity.

The gravitational acceleration at r produced by the star is

$$g = \frac{GM(r)}{r^2} \quad \text{(radially inwards)}, \tag{7}$$

and the gravitational force acting on the parcel of gas (of mass m) is $F = mg$. In order to balance the net pressure force, we must have

$$-\frac{dP}{dr}\Delta r\, A = m\,\frac{GM(r)}{r^2}, \tag{8}$$

and dividing by m gives the **equation of hydrostatic equilibrium**:

$$\boxed{-\frac{1}{\rho}\frac{dP}{dr} = \frac{GM(r)}{r^2}} \tag{9}$$

(using $\rho(r) = m/(\Delta r\, A)$, because $\Delta r\, A$ is the volume of the gas parcel).

This equation can tell us about conditions in the invisible stellar interior. As an example, we can estimate the pressure at the centre of the Sun, P_C. As a rough approximation, we can assume a typical pressure gradient $dP/dr \simeq P_C/R$ (i.e. the surface pressure is negligibly small). Hydrostatic equilibrium relates this typical gradient to a typical density and a typical acceleration. We take the former to be the mean density, $\bar{\rho}$, and the latter to be the surface acceleration, GM/R^2. This gives

$$P_C \simeq GM\bar{\rho}/R = 3GM^2/4\pi R^4. \tag{10}$$

A more careful argument shows that this is an inequality. If we divide the equation of hydrostatic equilibrium by the equation of mass conservation, we get

$$-\frac{dP/dr}{dM/dr} = -\frac{dP}{dM} = \frac{GM(r)}{4\pi r^4} > \frac{GM(r)}{4\pi R^4} \tag{11}$$

Integrating dM from 0 to M then gives

$$P_C > GM^2/8\pi R^4, \tag{12}$$

which is almost the previous result. This pressure is $4.5 \times 10^{13}\,\mathrm{N\,m^{-2}}$, or $10^{8.6}$ times atmospheric pressure.

What is the corresponding temperature? We assume the **perfect gas law**,

$$P = \rho kT/\bar{m}, \tag{13}$$

so the central temperature is

$$T_C = \bar{m}P_C/k\rho_C. \tag{14}$$

If we assume that the central density is of order the mean density, this suggests a minimum temperature of

$$kT_C \gtrsim \frac{G\bar{m}M}{R} \simeq 1.2 \times 10^7 \text{ K}. \tag{15}$$

(taking $\bar{m} = m_p/2$, and discarding factors of order unity). In other words, the typical thermal energy is of order the gravitational binding energy. This approximate equality between kinetic and potential energies is very common in self-gravitating structures, and is known as the **virial theorem**. This temperature has been deduced using some rather dubious assumptions, but the final figure for the temperature is not so far wrong: the correct central temperature for the Sun is 1.6×10^7 K.

Now, we would like to get an equation for $L(r)$: the luminosity passing through each layer r in the star. [$L(R)$ is the luminosity emanating from the stellar surface, which is the star's luminosity]. To do this, we must consider the source of a star's energy.

2 Energy Generation in stars

The Sun loses energy at the rate $L_\odot = 3.8 \times 10^{26}$ W. How long would it take the Sun to use up all of its thermal energy? If each particle in the Sun carries a thermal energy $3kT/2$ on average (by **equipartition of energy**), then we have for the total thermal energy of these particles

$$U_{\text{thermal}} \simeq (M_\odot/\bar{m}) \times 3kT/2 \simeq (M_\odot/\bar{m}) \times \frac{3GM_\odot\bar{m}}{2R_\odot} = \frac{3GM_\odot^2}{2R_\odot} \simeq 6 \times 10^{41} \text{ J}. \tag{16}$$

How long does it take the Sun to radiate away all this thermal energy?

$$t_{\text{thermal}} = \frac{U_{\text{thermal}}}{L_\odot} = \frac{6 \times 10^{41}\text{ J}}{3.8 \times 10^{26}\text{ W}} \simeq 1.6 \times 10^{15}\,\text{s} \simeq 50\,\text{Myr}. \tag{17}$$

This is much less than the age of the Earth, which is 4.56 Gyr. This was a fundamental problem recognised already by the end of the 19th century by Kelvin & Helmholtz (sometimes t_{thermal} is called the **Kelvin-Helmholtz time**). It tells us that the Sun must have some additional energy-generating mechanism. One of the great achievements of 20th Century physics was to work out what powers the stars. It just comes down to $E = mc^2$.

As soon as nuclear reactions were discovered in the early 20th Century, it was clear that they were a much greater potential source of energy than any chemical reaction, and were thus a plausible source of energy for a star. Consider **fusing** 4 protons (from the ionised hydrogen in the star) into a helium nucleus:

$$4p \to {}^4\mathrm{He} + \mathrm{Energy}\ Q_{\mathrm{eff}} \tag{18}$$

(Note that ^{4}He consists of 2 protons and 2 neutrons, so 2 of the protons would need to convert into neutrons plus positrons in this process.) Now, the measured mass of ^{4}He is *less* than the total mass of the individual 4 protons (this is true even though the mass of a neutron is slightly more than the mass of a proton).

We know then that some mass Δm was lost, corresponding to an energy $Q_{\mathrm{eff}} = \Delta mc^2$. Energy must be conserved in the fusion process, so this much energy is carried away by radiation: that's what Q_{eff} represents. Physically, there is a small but negative **binding energy** holding the 2 protons and 2 neutrons together in the ^{4}He nucleus. This binding energy represents the amount of energy needed to cause the nucleus to undergo **fission** into its component parts. Quantitatively, we find $\Delta m \simeq 0.007m(^4\mathrm{He})$, so that nearly 1% of the mass of the protons is converted into energy.

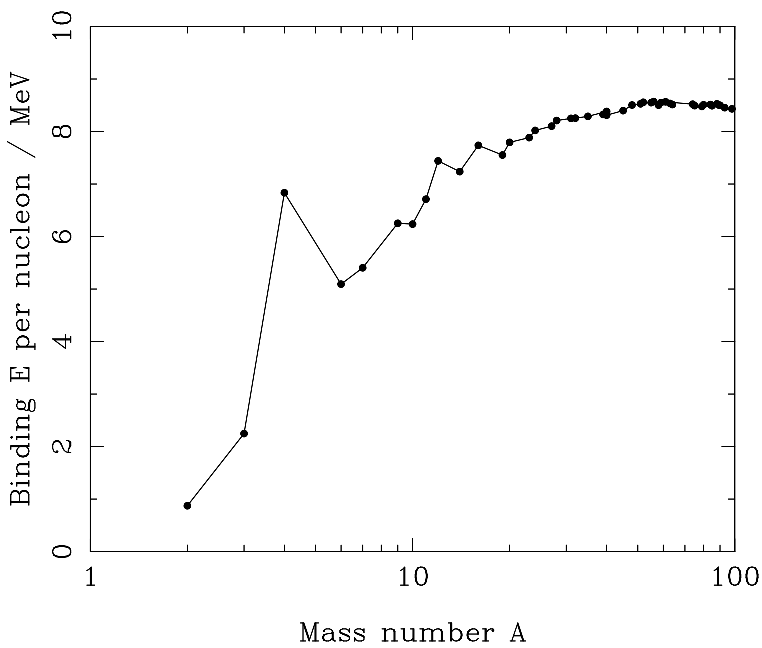

Figure 1: As we move to heavier nuclei, the inter-nuclear forces cause them to be more strongly bound. The best way of quantifying the relative strengths of these bonds is via the binding energy per nucleon (as measured by the mass of the nucleus compared to its component parts). In this plot (showing the most abundant isotopes of each element), two things stand out: (1) the relatively strong binding of ^{4}He; (2) the maximum at Fe ($Z = 26$, $A = 56$), which is the most stable element.

This is all very well, but fusion reactions will only happen if the protons can approach within the range of the strong nuclear interaction (about 10^{-15} m). The protons are electrically charged and repel each other, so it is not clear whether fusion will happen in practice.

2.1 Classical view vs quantum tunneling

Start with a classical argument (which will turn out to be wrong). Two protons at large separation have a total energy that is roughly their kinetic energy, $E_{\mathrm{tot}} \simeq E_K$, because their Coulomb energy decays with separation: $E_C = e^2/4\pi\epsilon_0 r$. To fuse, they must be brought together into the very small space of a nucleus: $r_N \simeq 10^{-15}$ m.

If they were initially given just enough kinetic energy that the Coulomb repulsion brings them exactly to rest as they reach a separation r_N, then their final total energy will just be electrostatic potential energy. Conservation of energy then says

$$E_{\rm tot}^{\rm init} = E_{\rm tot}^{\rm final} \quad \Rightarrow \quad E_K^{\rm init} = E_C^{\rm final} = \frac{e^2}{4\pi\epsilon_0 r_N} \simeq 2 \times 10^{-13}\,{\rm J}. \tag{19}$$

If the protons have a temperature T, then the mean kinetic energy of two protons will be $E_K = 2 \times (3kT/2)$, and the required temperature will be

$$T = \frac{E_C}{3k} \simeq 10^{10}\,{\rm K}. \tag{20}$$

However, this is *much* hotter than the Sun's centre. This was a major stumbling block to advocates of nuclear fusion as the energy source in stars. Still, some weren't dissuaded, like Sir Arthur Eddington who said it must somehow work at the Sun's temperature, '...we tell him [the critic] to go and find a hotter place!'. The answer was recognised by George Gamow, and lies in the intrinsically non-classical phenomenon known as **quantum tunneling**. Essentially, there is a finite probability for protons to fuse at any given energy E, which is proportional to $e^{-(E_G/E)^{1/2}}$. This is called **penetration factor**, and the energy E_G is called **Gamow energy**. This roughly means that particles with energies E_G and above should easily be able to tunnel through the Coulomb barrier and fuse.

The Gamow energy E_G is given by

$$E_G = (\pi\alpha Z_A Z_B)^2\ 2m_r c^2 \tag{21}$$

where Z_A and Z_B are the charges of two fusing particles A and B, $m_r = m_A m_B/(m_A + m_B)$ is the reduced mass, and $\alpha \equiv e^2/(4\pi\epsilon_0 \hbar c) \simeq 1/137$. Note that the Gamow energy is nearly two orders of magnitude higher for the fusion of a proton with a carbon nucleus than for the fusion of two protons.

2.2 Nuclear fusion

In order to determine the nuclear fusion rate, R_{AB}, we need to think about the available number of particles at a given energy. In thermal equilibrium the particle energies are distributed according to the Boltzmann factor,

$$P_{\rm Boltz} \propto \exp(-E/kT), \tag{22}$$

where T is the temperature of the gas. So, the total probability for particles to fuse depends on particle energies according to

$$\boxed{R_{AB} \propto P_{\rm tot} = P_{\rm Boltz} \times P_{\rm tunnel} \approx \exp\left[-E/kT - (E_G/E)^{1/2}\right]} \tag{23}$$

This rate peaks at about $E = E_0 = (kT)^{2/3} E_G^{1/3}/2^{2/3}$. For the fusion of two protons, $E_G = 493$ keV, whereas in the Sun's core ($T = 1.6 \times 10^7$ K) $kT = 1.4$ keV, so that $E_0 \simeq 6.2\,{\rm keV} \simeq 4.4kT$. Notice that this automatically gives a strong temperature dependence of the reaction rate: fusion in the Sun involves protons that are on the rare exponential tail of the probability distribution, so that a small change in temperature causes a large change in the reaction rate.

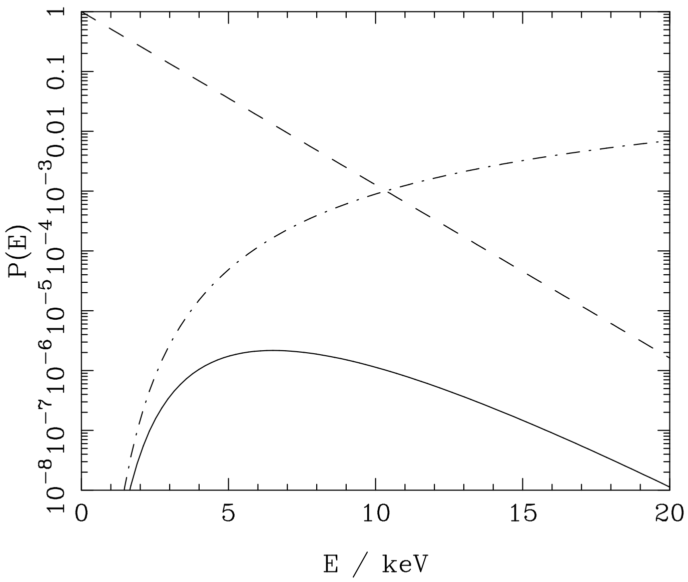

Figure 2: The probability that a proton of energy E in the Sun will be able to fuse scales as the Boltzmann thermal probability distribution (dashed) times the tunneling probability (dot-dashed). The product (solid line) peaks in the Sun at about 6 keV, or 4 times the typical thermal energy.

3 Nuclear Reactions

Nuclear reactions fusing $4p \rightarrow {}^4\text{He}$ occur by two principal mechanisms: (1) The PP chain; (2) The CNO Cycle.

3.1 PP chain (for stars with $M < M_\odot$)

The dominant reaction branch gives deuterium initially (^{2}H), which then combines with a third proton to produce ^{3}He:

$$\begin{aligned} p + p &\rightarrow d + e^+ + \nu_e \\ p + d &\rightarrow {}^3\text{He} + \gamma \end{aligned} \tag{24}$$

There are then three possible continuations for the ^{3}He:

$$\begin{aligned}
\text{Branch 1 :} \quad {}^3\text{He} + {}^3\text{He} &\rightarrow {}^4\text{He} + 2p \\
\\
\text{Branch 2 :} \quad {}^3\text{He} + {}^4\text{He} &\rightarrow {}^7\text{Be} + \gamma \\
e^- + {}^7\text{Be} &\rightarrow {}^7\text{Li} + \nu_e \\
p + {}^7\text{Li} &\rightarrow {}^4\text{He} + {}^4\text{He} \\
\\
\text{Branch 3 :} \quad {}^3\text{He} + {}^4\text{He} &\rightarrow {}^7\text{Be} + \gamma \\
p + {}^7\text{Be} &\rightarrow {}^8\text{B} + \gamma \\
{}^8\text{B} &\rightarrow {}^8\text{Be}^* + e^+ + \nu_e \\
{}^8\text{Be}^* &\rightarrow {}^4\text{He} + {}^4\text{He}
\end{aligned} \tag{25}$$

In each case the sum total gives

$$4p \to {}^4\text{He} + Q_{\text{eff}} + 2e^+ + 2\nu_e, \tag{26}$$

where Q_{eff} includes the energy lost directly as photons and by the annihilation of $2e^+$ with ambient electrons in the plasma; it does not include the energy carried away as neutrinos. According to the standard model, Branch 1 occurs about 85% of the time in the sun, and has $Q_{\text{eff}} = 26.2$MeV, Branch 2 occurs about 15% of time, with $Q_{\text{eff}} = 25.2$MeV, and Branch 3 occurs about 0.02% of the time, with $Q_{\text{eff}} = 19.1$MeV.

For T close to 10^7 K, the **energy generation rate per unit mass**, ϵ, is given approximately by

$$\epsilon_{pp} \simeq 1.07 \times 10^{-8} \rho X^2 T_7^4 \text{ W kg}^{-1}, \tag{27}$$

where X is the mass fraction in H and $T_7 \equiv T/10^7$ K.

3.2 CNO Cycle ($M > M_\odot$)

This is a more complex chain of nuclear reactions involving proton capture and decay:

$$\begin{aligned}
p + {}^{12}\text{C} &\to {}^{13}\text{N} + \gamma \\
{}^{13}\text{N} &\to {}^{13}\text{C} + e^+ + \nu_e \\
p + {}^{13}\text{C} &\to {}^{14}\text{N} + \gamma \\
p + {}^{14}\text{N} &\to {}^{15}\text{O} + \gamma \\
{}^{15}\text{O} &\to {}^{15}\text{N} + e^+ + \nu_e \\
p + {}^{15}\text{N} &\to {}^{12}\text{C} + {}^4\text{He}
\end{aligned} \tag{28}$$

The original ^{12}C nucleus has gone back into circulation by the end, resulting in the cycle – it therefore acts as a catalyst for the nuclear generation of Helium. As for the pp chain, the net result is

$$4p \to {}^4\text{He} + Q_{\text{eff}}, \tag{29}$$

where now $Q_{\text{eff}} = 23.8$ MeV (the neutrinos would increase this by 6%). For T close to 10^7 K, the energy generation rate is approximately given by

$$\epsilon_{CNO} \simeq 6.54 \times 10^{-11} \rho X X_{CNO} T_7^{20} \text{ W kg}^{-1}, \tag{30}$$

where X_{CNO} is the total mass fraction in C, N and O. The stronger temperature dependence of the CNO cycle arises because of the much higher Coulomb barriers in the fusion reactions.

Because of their different temperature dependences, the pp chain is the dominant energy generation mechanism in low mass (cooler) stars, while the CNO cycle dominates in more massive (hotter) stars. About 98–99% of the energy production in the sun is through the pp chain, but stars only 20% more massive are dominated by the CNO cycle.

3.3 Equation of energy generation

How much energy will be generated in a shell of thickness Δr? Since ϵ gives the energy-generation rate per unit mass (in $\mathrm{W\,kg^{-1}}$), the rate of energy generation from the shell per unit volume is $\epsilon\rho(r)$, where $\rho(r)$ is the density of the star in the shell at radius r. The volume of the shell is $4\pi r^2\Delta r$, so the rate of energy generation in the shell that contributes to the total luminosity of the star (in W) is

$$\Delta L = (4\pi r^2 \Delta r)\epsilon\rho. \tag{31}$$

Taking the limit of infinitesimal Δr, we get the **equation of energy generation**:

$$\boxed{\frac{dL}{dr} = 4\pi r^2 \epsilon\rho.} \tag{32}$$

Here, ϵ is in general $\epsilon = \epsilon_{pp} + \epsilon_{CNO}$ since both the pp-chain and the CNO cycle will contribute to the total energy generation rate at some level. Most of the energy is generated in the central core of the star where the temperature is highest.

4 Radiative diffusion

How does the energy generated in the central core of a star escape to the star's surface? The high density of atoms and ions in a star act as efficient scatterers of photons. A typical average distance a photon in the core of a star can travel before scattering – the **mean free path** – is 1 mm. The very short mean free path of a photon has two important consequences: (1) The radiation field takes on a blackbody spectral shape; (2) The radiation leaks out slowly by radiative diffusion.

The diffusion equation for particles is given by:

$$J = -D\, dn/dx, \tag{33}$$

where J is the **flux density** of particles (number of particles crossing a unit area per unit time), n is the number density of the particles, and D is the diffusion coefficient. Statistical mechanics gives

$$D = \ell v/3, \tag{34}$$

where ℓ is the mean free path for particles moving at velocity v. The same relation will hold for photons of number density n_ν where the subscript of 'ν' is added to allow for the spectrum of the radiation (photons of different frequencies ν have different number densities). We then have for the diffusion equation of photons of frequency ν

$$J_\nu = -D_\nu dn_\nu/dr \tag{35}$$

for diffusion in the radial direction outwards from the star's centre. Here

$$D_\nu = \ell_\nu c/3 \tag{36}$$

is the diffusion constant for the photons, all of which move at the speed of light c between scatterers, but have a mean free path ℓ_ν that in general will depend on ν.

The energy flux density is then given by

$$f_\nu = (h\nu)J_\nu = -D_\nu d(h\nu n_\nu)/dr = -D_\nu dU_\nu/dr, \tag{37}$$

where each photon of frequency ν has energy $h\nu$, and the energy density of such photons is

$$U_\nu = h\nu n_\nu. \tag{38}$$

To get the total energy flux passing through a shell at distance r, we need to integrate over all frequencies:

$$F = \int_0^\infty f_\nu \, d\nu = -\int_0^\infty D_\nu \, dU_\nu/dr \, d\nu \equiv -\bar{D} \, dU/dr \tag{39}$$

where $U = \int U_\nu \, d\nu$ is the total energy density of radiation field, and $\bar{D}$ is a frequency-averaged diffusion coefficient.

By convention, we define an **opacity** κ by

$$\boxed{\kappa = \frac{c}{3\rho\bar{D}}.} \tag{40}$$

The opacity has units of $\mathrm{m^2kg^{-1}}$. It describes the average scattering cross section of a photon on passing through 1 kg of material responsible for the scattering. So we can write

$$F = -\frac{c}{3\rho\kappa}\frac{dU}{dr}. \tag{41}$$

Since the radiation is black body, $U = aT^4$, where T is the temperature of the stellar material. So we finally obtain the **equation of radiative diffusion**:

$$\boxed{F = \frac{L}{4\pi r^2} = -\frac{4ac}{3\rho\kappa} T^3 \frac{dT}{dr}.} \tag{42}$$

At low temperatures, the gas is only partially ionized. The opacity is then dominated by bound-free absorption. At higher temperatures, when the ionization is nearly complete, the opacity is dominated by free-free absorption. The resulting frequency-averaged opacity depends on density and temperature according to (for bound-free transitions)

$$\kappa \propto \rho T^{-3.5} \tag{43}$$

This is known as **Kramer's Law**.

Scattering by electrons is also always present. If σ_T denotes the Thomson cross section for scattering of photons by free electrons, this gives an opacity of

$$\kappa_{\mathrm{es}} = \frac{n_e\sigma_T}{\rho} = (1+X)\frac{\sigma_T}{2m_H} \simeq 0.02(1+X)\ \mathrm{m^2kg^{-1}}, \tag{44}$$

where n_e is the number density of electrons, and X is the mass fraction of hydrogen in the star. Thomson scattering is the dominant source of opacity in regions of low density and high temperature.

To give some indicative values of the opacity and mean free path in stars, for material of solar abundances, at $\rho = 10^4\,\mathrm{kg\,m^{-3}}$ and $T = 2 \times 10^6$ K, $\kappa \simeq 10\ \mathrm{m^2kg^{-1}}$. This gives a mean free path of $\ell = 1/\kappa\rho \simeq 0.1$ mm. At a higher temperature $T = 10^7$ K, the opacity decreases to $\kappa \simeq 0.1\ \mathrm{m^2kg^{-1}}$ and the photon mean free path increases to $\ell \simeq 1$ mm.

5 Summary of the equations of stellar structure

Define:

$\rho(r)$ = mass density at radius r
$T(r)$ = temperature at radius r
$P(r)$ = pressure at radius r
$M(r)$ = mass within radius r
$L(r)$ = luminosity escaping through a surface at r
$F(r)$ = flux of radiation escaping through a surface at r: $F(r) = L(r)/(4\pi r^2)$
$\epsilon(r)$ = nuclear energy generation rate per unit mass at r
$\kappa(r)$ = opacity of stellar material at r

$$
\begin{aligned}
&\text{(1) Equation of Continuity :} && \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \\
&\text{(2) Equation of Hydrostatic Equilibrium :} && \frac{1}{\rho}\frac{dP}{dr} = -\frac{GM(r)}{r^2} \\
&\text{(3) Equation of Energy Generation :} && \frac{dL}{dr} = 4\pi r^2 \epsilon\rho \\
&\text{(4) Equation of Radiative Diffusion :} && F = \frac{L}{4\pi r^2} = -\frac{4ac}{3\rho\kappa}\, T^3\, \frac{dT}{dr}
\end{aligned}
\tag{45}
$$

We would like to solve these subject to the perfect gas law $P(r) = [\rho(r)/\bar{m}]\, kT(r)$ and simple power-law scalings for the nuclear energy generation rate and the opacity:

$$\epsilon = \epsilon_0 \rho T^\alpha; \qquad \kappa = \kappa_0 \rho^\beta / T^\gamma. \tag{46}$$