Radiation and Matter

Tutorial Sheet 1 Solutions

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1. We would like to gauge transform our fields $\mathbf{A}$ and ϕ into new fields $\mathbf{A}'$ and ϕ' so that

$$\nabla.\mathbf{A}' + c^{-2}\partial\phi'/\partial t = 0\,.$$

Recall that the gauge transformations (transformations that leave the *physics* unhanged are:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi\,,$$

$$\phi' = \phi - \dot{\chi}\,.$$

The field χ is our gauge field; its value has no effect on the physics, only the mathematics. If we replace our transformed (primed) fields by their original values, we see

$$\nabla^2\chi - c^{-2}\partial^2\chi/\partial t^2 = -\left(\nabla.\mathbf{A} + c^{-2}\partial\phi/\partial t\right)\,.$$

The right-hand side is our original expression that is *not* equal to zero, and acts as a source function for the wave-equation for χ appearing on the left-hand side. If we could solve this wave-equation for χ, we could determine what χ to use so that our transformed expression is equal to zero. Since the question only asks for a *prescription*, there is no need to attempt to write down a general solution, only to state that the prescription is to do just that.

2. Like all proofs, its difficulty varies tremendously based on what you care to assume. The question is not specific on this, so you should probably treat the question as an opportunity to explore how relativity and electromagnetism are related, rather than a precisely worded task.

The Lorentz gauge : $\nabla \cdot \mathbf{A} + c^{-2}\partial\phi/\partial t = 0$.

Which can be re-written as two four vectors dotted together:

$$\begin{pmatrix} c^{-1}\partial/\partial t \\ -\nabla \end{pmatrix} \cdot \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix} = 0$$

Now you can either:

- Assume that these are both four vectors[1] and so the expression is Lorentz invariant.
- Go through the pain of Lorentz transforming each four vector in the equation and showing that the whole thing is indeed invariant (which you know it should be anyway from the previous point).

Well if you choose to Lorentz transform your equation:

$$\begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t'} \\ -\frac{\partial}{\partial x'} \\ -\frac{\partial}{\partial y'} \\ -\frac{\partial}{\partial z'} \end{pmatrix} \cdot \begin{pmatrix} \frac{\phi'}{c} \\ A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\phi}{c} \\ A_x \\ A_y \\ A_z \end{pmatrix} = 0$$

$$= \begin{pmatrix} \frac{\gamma}{c}\frac{\partial}{\partial t} + \gamma\beta\frac{\partial}{\partial x} \\ \frac{-\beta\gamma}{c}\frac{\partial}{\partial t} - \gamma\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \gamma\frac{\phi}{c} - \gamma\beta A_x \\ -\beta\gamma\frac{\phi}{c} + \gamma A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= \frac{\gamma^2}{c^2}\frac{\partial\phi}{\partial t} - \frac{\gamma^2\beta}{c}\frac{\partial A_x}{\partial t} + \frac{\gamma^2\beta}{c}\frac{\partial\phi}{\partial x} - \gamma^2\beta^2\frac{\partial A_x}{\partial x} - \frac{\gamma^2\beta^2}{c^2}\frac{\partial\phi}{\partial t} + \frac{\gamma^2\beta}{c}\frac{\partial A_x}{\partial t} - \frac{\gamma^2\beta}{c}\frac{\partial\phi}{\partial x} + \gamma^2\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}$$

$$= \gamma^2(1-\beta^2)\frac{\partial\phi}{c\partial t} + \gamma^2(1-\beta^2)\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}$$

[1] A four vector is something whose magnitude is unchanged under Lorentz transformation

$$= \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t'} \\ -\frac{\partial}{\partial x'} \\ -\frac{\partial}{\partial y'} \\ -\frac{\partial}{\partial z'} \end{pmatrix} \cdot \begin{pmatrix} \frac{\phi'}{c} \\ A'_x \\ A'_y \\ A'_z \end{pmatrix}$$

Which is our original equation, written in four vector form. Hence it is Lorentz covariant.

We can also write the expression in index notation:

$$\frac{\partial}{\partial x^\mu} A^\mu = \partial_\mu A^\mu = \partial_\mu \partial^\mu \chi \,,$$

where we've adopted a different gauge to be as general as possible.

Advanced and optional (for those of you studying relativity):

You might like to see how little you can get away with assuming. It is actually possible to *prove* the four-vector ∂_μ is a four-vector. Recall that the definition of a four vector is that **if** we have two coordinate systems (primed and umprimed) related in the following way (remember to sum over repeated indices):

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

where $\Lambda^\mu{}_\nu$ are constants, **then** a four vector V^μ transforms as

$$V'^\mu = \Lambda^\mu{}_\nu A^\nu \,.$$

To be precise we should call V'^μ a *contravariant* four-vector, to distinguish it from a *covariant* four-vector that has a similar but different rule:

$$V'_\mu = \Lambda_\mu{}^\nu A_\nu \,.$$

Note the location of the index is now below the A symbol (a useful mnemonic: *co* rhymes wih *below*). The matrix components $\Lambda_\mu{}^\nu$ are still constants, but is the inverse to the previous matrix:

$$\Lambda_\alpha{}^\gamma \Lambda^\alpha{}_\beta = \delta^\gamma{}_\beta \,.$$

Now consider the derivative four-vector. Using the chain rule,

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta} \,.$$

Returning to our coordinate transformation,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

we can multiply by $\Lambda_{\alpha}^{\gamma}$ on both sides, and take partial derivatives to yield,

$$\frac{\partial x^{\beta}}{\partial x'^{\alpha}} = \Lambda_{\alpha}^{\ \beta} \, .$$

We can substitute this into our earlier equation for the gradient,

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}} = \Lambda_{\alpha}^{\ \beta} \frac{\partial}{\partial x^{\beta}} \, ,$$

which is the definition of a covariant four-vector.

Going back to our gauge condition, the right-hand side is $\partial_{\mu}\partial^{\mu}$ (the contraction of two four-vectors is a scalar) multiplying χ. As χ is (we will assert) a scalar, and a scalar multiplying a scalar gives another scalar, then we have the following situation:

$$\partial_{\mu} A^{\mu} = f$$

where now we know ∂_{μ} to be a four-vector and f to be a scalar. From this we can prove A^{μ} is also a four-vector, by noting that f doesn't change if we go to our primed coordinate system:

$$\partial_{\mu} A^{\mu} = f = \partial'_{\mu} A'^{\mu} \, .$$

Since ∂'_{μ} is a four vector, we know how it relates to its version in the unprimed coordinate system:

$$\partial_{\mu} A^{\mu} = \Lambda^{\nu}{}_{\mu} \partial_{\nu} A'^{\mu} \, .$$

And now with a little rearrangemnt:

$$\partial_{\mu} (A^{\mu} - \Lambda^{\nu}{}_{\mu} \partial_{\nu} A'^{\mu}) = 0 \, .$$

Either $\partial_{\mu} A^{\mu}$ is zero (which is not, in general, true - only in some gauges is this the case) or we must have:

$$(A^{\mu} - \Lambda^{\nu}{}_{\mu} \partial_{\nu} A'^{\mu}) = 0$$

which implies $A^{\mu} = \Lambda^{\nu}{}_{\mu} \partial_{\nu} A'^{\mu}$, precisely the definition of a four vector.

This proof is actually an example of the **Quotient Law**, which generalises the result to expressions with as many indices as you care to use.

3. We'd like to go from our old gauge where $\nabla\dot{\mathbf{A}} \neq 0$ to a new one where $\nabla \cdot \mathbf{A}' = 0$. Just like in Question 1, we replace our new (primed) gauge fields by their old (unprimed) ones through the gauge transformation, and we get

$$\nabla \cdot \mathbf{A} + \nabla^2 \chi = 0,$$

which provides our expression for χ.

Remember that ϕ also gets changed,

$$\phi' = \phi - \dot{\chi},$$

and we should (according to the question!) see our new ϕ' obey Poisson's equation (sourced by the charge distribution). The easiest way to get at the charge distribution is via a Maxwell equation:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0\,.$$

Inserting the definition of $\mathbf{E} = -\nabla\phi' - \partial\mathbf{A}'/\partial t$ into our expression,

$$\nabla \cdot \mathbf{E} = -\nabla^2\phi' - \nabla.\left(\partial\mathbf{A}'/\partial t\right) = \rho/\epsilon_0\,.$$

But it doesn't matter which acts first on $\mathbf{A}'$, the ∇ or the $\partial/\partial t$. So we can swap the order and use our gauge condition $\nabla.\mathbf{A}' = 0$, removing the term and giving us Poisson's equation. Note that there is no retardation involved; the potential ϕ' is determined by ρ instantaneously and everywhere.

In the wave zone, by definition $\mathbf{A}'$ is a function of $t - \mathbf{r}.\hat{\mathbf{n}}/c$. Using the Mathematical Appendix (or fairly simply derived), we can replace ∇ by $-(\hat{\mathbf{n}}/c)(\partial/\partial t)$:

$$\nabla \cdot \mathbf{A}' = -\left(\frac{\hat{\mathbf{n}}}{c}\right) \cdot \frac{\partial \mathbf{A}'}{\partial t} = 0$$

Thus $\hat{\mathbf{n}} \cdot \dot{\mathbf{A}}' = 0$ as required.

4. Our new gauge now needs both $\nabla.\mathbf{A}' = 0$ (the same as the previous question) *and* $\phi' = 0$. We know immediately that the condition is

$$\nabla.\mathbf{A} + \nabla^2\chi = 0\,,$$

from our previous question. We also found that

$$-\nabla^2\phi' = \rho/\epsilon_0\,.$$

Since we're also demanding $\phi' = 0$, then there can be no charges present at all.

5a. Evaluate:

$$\int f(x)\delta(g(x))\,dx$$

So this is a problem because you have the delta function not of a single variable, but a function. Invert the g(x) equation to put the $g(x) \to g$ as the variable in the integral.

$$= \int f(x(g))\delta(g)\frac{dx}{dg}\,dg$$

Now you know the solution to this integral, it just picks out the value of the integrand at $g = 0$ ...

$$= (f(x(g))\frac{dx}{dg})|_{g=0} = \frac{f(x(g=0))}{\frac{dg}{dx}|_{g=0}}$$

5b. Evaluate:

$$\int \sin^3\alpha\,d\alpha d\phi = \frac{8\pi}{3}$$

Write $\sin^2\alpha = 1 - \cos^2\alpha$ and take out the ϕ integral:

$$= \int_0^{2\pi} d\phi \int \sin\alpha(1-\cos^2\alpha)\,d\alpha = 2\pi\int \sin\alpha - \sin\alpha\cos^2\alpha\,d\alpha$$

$$= 2\pi\left[-\cos\alpha + \frac{\cos^3\alpha}{3}\right]_0^\pi = \frac{8\pi}{3}$$

6. Maxwell's equations:

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \frac{\partial \mathbf{E}\epsilon_0}{\partial t}) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In this case we are in free space, so there are no sources ($\rho = 0$) and therefore no current density ($\mathbf{j} = \mathbf{0}$).

For an electromagnetic wave then we have

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{c^2 \partial t} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and we know that $\mathbf{E}, \mathbf{B} = \mathbf{F}(t - \frac{\mathbf{r} \cdot \mathbf{n}}{c})$, which, from page 4 of the notes means we can rewrite the spatial derivatives as time derivatives. Which can be rewritten using a Maxwell equation to give the LHS:

$$\nabla \times \mathbf{B} = -\frac{\hat{\mathbf{n}}}{c} \times \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial \mathbf{E}}{c^2 \partial t}$$

Removing the time dependence gives $\mathbf{B} \times \hat{\mathbf{n}} = \frac{\mathbf{E}}{c}$ and using the equivalent $\mathbf{E}$ equation you get $\hat{\mathbf{n}} \times \mathbf{E} = c\mathbf{B}$. Hence $\mathbf{E}, \mathbf{B}$ and $\hat{\mathbf{n}}$ (in the $\mathbf{k}$ direction) are perpendicular and form a right handed set.

Because you know that the three vectors are perpendicular you can write $\mathbf{B} \times \hat{\mathbf{n}} = |\mathbf{B}||\hat{\mathbf{n}}| \sin \pi/2 = |\mathbf{B}|$ in the direction of $\mathbf{E}$. And using the equation derived above this implies $|\mathbf{E}| = c|\mathbf{B}|$.

7. This is an exercise to help you explore the different definitions involved.

Consider a beam of radiation, filled with photons of number density n. Imagine the beam striking a flat target of area A. In a time T, $AncT$ photons pass through the target area. Each photon has energy $\hbar\omega$, so the energy per unit time per unit area (the definition of flux) passing through the target is $S = cn\hbar\omega$.

Momentum density is the momentum p carried by each photon, multiplied by the number density, pn. Since photons have a momentum $\hbar\omega/c$, then

$$S = nc\hbar\omega = nc^2(\hbar\omega/c) = \text{momentum density} \times \mathrm{c}^2\,.$$

8a. The mathematical appendix has some rules that make this question much easier. The particular rule states that ∇ acting upon a wave solution has the same effect as operating with $(-\hat{\mathbf{n}}/c)\,(\partial/\partial t)$. Replacing the two ∇ operators like this naturally leads to an identity, showing the wave solution satisfies this equation.

8b. and 8c. The ∇ operating on the exponential brings down $-i\mathbf{k}$, so doing so twice just gives k^2. For the sin function, the first effect is to give $-\mathbf{k}$ times cos, while operating again gives k^2. Very similar results emerge from the time derivatives, showing these satisfy the wave equation too, provided $w = kc$.

9. Non-relativistic + small circle tells us that we can use the Larmor formula to calculate the power lost from the charge as it is accelerated in a circle:

$$P_{larmor} = \frac{dE}{dt} = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c^3}\dot{v}^2$$

So we need to find the acceleration of the charge and we will be away. The magnitude of acceleration in a circle is given by $a = \frac{v^2}{r}$.

Hence the rate of emission of energy is:

$$P_{larmor} = \frac{dE}{dt} = \frac{2}{3}\frac{q^2 v^4}{4\pi\epsilon_0 c^3 r^2}$$

For a Bohr orbit in an atom the total energy is given by the kinetic plus the electrostatic:

$$E_{Bohrorbit} = \frac{mv^2}{2} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{e^2}{8\pi\epsilon_0 r}$$

where the final form is obtained by using the fact that the centripetal force and electrostatic force must balance for the electron to remain in orbit.

So we now have:

$$\frac{dE}{dt} = \frac{2}{3}\frac{q^2 v}{4\pi\epsilon_0 r^2}\left(\frac{v}{c}\right)^3 \quad E(r(t)) = -\frac{e^2}{8\pi\epsilon_0 r}$$

Which is the rate of loosing energy per second and the total energy, so we can get a rough timescale for decay:

$$\text{lifetime} = \frac{\text{inital energy}}{\text{energy lost per second}} = \frac{E_{orbit}}{P} = \frac{3}{4}\frac{r}{v}\left(\frac{c}{v}\right)^3 \simeq 10^{-10}s$$

Where the final number is calculated using the fact that the Bohr atom also has quantised angular momentum of $L = mvr = \hbar$ in the ground state, and $v/c = 1/137$.

10. Here the question asks for power, however this would give you one number which is the flux intregrated over all angles. Hence consider the flux you would observe at a given position. From the notes we know that $\mathbf{S} \propto \dot{\mathbf{A}}_\perp$ (p5) and in the non-relativistic limit in the wave zone $\mathbf{A} \propto \mathbf{v} \Rightarrow \dot{\mathbf{A}}_\perp \propto \dot{\mathbf{v}}_\perp$. Hence $\mathbf{S} \propto \dot{\mathbf{v}}_\perp^2$ - the Poynting vector, which represents the energy flux, is proportional to the rate of change of the perpendicular component of the acceleration.

Now draw a diagram showing the acceleration vector, a line of sight and the angle between them. If you find the component of the acceleration perpendicular to the line of sight, this is then proportional to the energy flux in this direction.

a

In the first case, assume the particle is oscillating up and down the z-axis and hence your acceleration vector is also along the z-axis. Alpha in **this case** is equivalent to θ in polar coordinates, and if you then find the component of the acceleration perpendicular to $\hat{\mathbf{n}}$ (your line of sight) you have:

$$\dot{\mathbf{v}}_\perp^2 \propto \sin\alpha$$

which gives you:

$$\mathbf{S} \propto \sin^2\alpha$$

So there is no flux looking down onto the particle oscillating, and maximum flux when looking perpendicular. The resultant energy flux distribution pattern is a doughnut, where here the surface represents the $\mathbf{S}$ vector.

b

This is slightly more complicated, but you can see just from considering different lines of sight and the perpendicular component of the acceleration to this, that there will be no direction where there is zero flux.

Have the particle oscillating in the y, x plane with acceleration vector $\mathbf{a}$ and choose the line of sight (in direction $\hat{\mathbf{n}}$) to be in the z, y plane for simplicity. Hence:

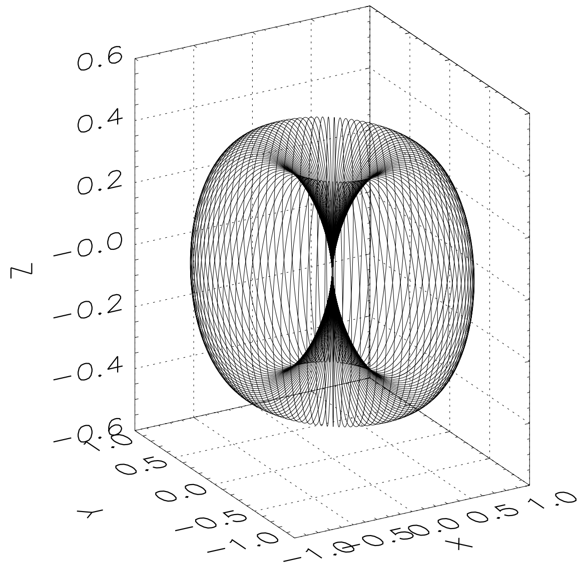

Figure 1: The polar diagram for part **a**. The charge is osciallating up and down the z-axis here.

$$\mathbf{a} = a(\cos\phi, \sin\phi, 0) \quad \hat{\mathbf{n}} = (0, \sin\theta, \cos\theta) \quad \phi = \omega t$$

To find the perpendicular component of the acceleration do some vector manipulation:

$$\mathbf{a}_\perp = \mathbf{a} - \mathbf{a}_\parallel = \mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$$

Substitute the vectors in to get:

$$\mathbf{S} \propto a^2(\cos^2\phi + \sin^2\phi\cos^4\theta + \sin^2\theta\cos^2\theta\sin^2\phi) = a^2(1+\cos^2\theta)$$

This shape is instantaneously the same at that from part **a**, however it changes as the particle circles around. To plot the polar diagram here, what you want is the **time averaged** flux vector. (Remember that $\phi = \omega t$). Hence the final equation above is obtained by time averaging, as $\langle\sin^2\omega t\rangle = 1/2$ etc..

Which is an egg shape with the long axis in the z direction.

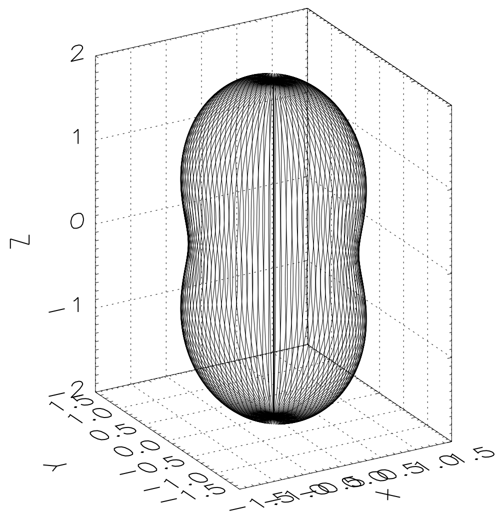

Figure 2: The polar diagram for part **b**, time averaged. The charge is circling around in the x-y plane here.

11. If we have a beam of photons, with number density n, aimed at an electron with cross-sectional area σ, we can recall from question **7** that per unit time, $c\sigma n$ photons will pass through the cross-section (i.e. be scattered). The total amount of energy scattered per unit time, the definition of power P, is given by $c\sigma n\hbar\omega$, which just equals the flux multiplied by cross-section (recall the definition of flux from question **7**). So $P = \sigma S$, or $\sigma = P/S$, which is in fact a general definition of cross-section.

So much for definitions; now turn to the physics. If an electron finds itself in the presence of a plane, monochromatic electromagnetic wave, it will experience a force. Assuming the motion is slow we can ignore the effects of the magnetic fields, so we only need to consider the force generated by the electric field, which produces an acceleration of $e\mathbf{E}/m$. The power radiated (working with a classical approximation, which is fine since huge numbers of photons will be emitted) is given by Larmor's formula. So,

$$P = \left(\frac{2}{3}\right)\frac{e^2}{4\pi\epsilon_0 c^3}\frac{e^2\mathbf{E^2}}{m^2}\,.$$

The flux S is given by the Poynting vector, which has a magnitude $\epsilon_0 c\mathbf{E^2}$. Plugging these into $\sigma = P/S$ gives the cross-section, which (after a little rearrangement) is what we were looking for. You have derived the Thomson scattering cross-section.

12. Classically

$$H = \int \frac{1}{2}\left(\epsilon_0|\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu_0}\right) dV$$

Choose a gauge such that $\phi = 0 \Rightarrow \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}$.
Now the vector potential can be written as a series of plane waves as:

$$\mathbf{A} = \sum_{\mathbf{k},\alpha} \kappa \hat{\mathbf{e}}_{\mathbf{k},\alpha}(a_{\mathbf{k},\alpha}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{\mathbf{k},\alpha}(t)e^{-i\mathbf{k}\cdot\mathbf{r}})$$

Where the time dependence of $a_{\mathbf{k},\alpha} \propto e^{-iwt}$ and $\kappa = \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}}$.
To find $|\mathbf{E}|^2$ differentiate wrt time...

$$\frac{\partial \mathbf{A}}{\partial t} = \sum_{\mathbf{k},\alpha} \kappa \hat{\mathbf{e}}_{\mathbf{k},\alpha}(-i\omega a_{\mathbf{k},\alpha}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + i\omega a^{\dagger}_{\mathbf{k},\alpha}(t)e^{-i\mathbf{k}\cdot\mathbf{r}})$$

Now take the modulus squared of this:

$$\left|\frac{\partial \mathbf{A}}{\partial t}\right|^2 =$$

$$\sum_{\mathbf{k}',\alpha'} \kappa' \hat{\mathbf{e}}_{\mathbf{k}',\alpha'}(-i\omega' a_{\mathbf{k}',\alpha'}(t)e^{i\mathbf{k}'\cdot\mathbf{r}} + i\omega' a^{\dagger}_{\mathbf{k}',\alpha'}(t)e^{-i\mathbf{k}'\cdot\mathbf{r}}) \sum_{\mathbf{k},\alpha} \kappa \hat{\mathbf{e}}_{\mathbf{k},\alpha}(-i\omega a_{\mathbf{k},\alpha}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + i\omega a^{\dagger}_{\mathbf{k},\alpha}(t)e^{-i\mathbf{k}\cdot\mathbf{r}})$$

By expanding out the equation above we are going to have lots of terms like this:

$$\int e^{i(\mathbf{k_1}-\mathbf{k_2})\cdot\mathbf{r}}\, dV$$

where the integral is over the total volume occupied by the modes. Now because the $\mathbf{k}$ vector is quantised, this integral is going to be zero unless $\mathbf{k_1} = \pm\mathbf{k_2}$ so that the integrand is unity. This results in only 4 terms remaining, 2 with $\mathbf{k} = \mathbf{k}$ and 2 with $\mathbf{k} = -\mathbf{k}$, and given that $a^{\dagger}_{-\mathbf{k},\alpha} = a_{\mathbf{k},\alpha}$ and $\hat{\mathbf{e}}_{\mathbf{k},\alpha} \cdot \hat{\mathbf{e}}_{-\mathbf{k},\alpha} = -1$ we finally end up with :

$$\left|\frac{\partial \mathbf{A}}{\partial t}\right|^2 = 2V\kappa^2\omega^2 \sum_{\mathbf{k},\alpha} (a_{\mathbf{k},\alpha} a^{\dagger}_{\mathbf{k},\alpha} + a^{\dagger}_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha})$$

For the magnetic field term note that (you need a vector identity here):

$$|\mathbf{B}|^2 = |\nabla \times \mathbf{A}|^2 = \left|\frac{\hat{\mathbf{n}}}{c} \times \frac{\partial \mathbf{A}}{\partial t}\right|^2 = \left(\frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}}{c^2}\right)\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 - \cancel{\left(\frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\hat{\mathbf{n}}}{c}\right)^2}$$

Hence the B field term is the same as the E field term we just calculated divided by c^2. Substitute into original expression to find the solution for the Hamiltonian.

13. The Poynting vector is related to the momentum density by:

$$\mathbf{K} = \frac{1}{c^2} \int \mathbf{S}\, dV$$

Using the same gauge as in question 12, the Poynting flux can be written as:

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0 = -\frac{\partial \mathbf{A}}{\partial t} \times (\nabla \times \mathbf{A})/\mu_0$$

$$= -\frac{\partial \mathbf{A}}{\partial t} \times \left(-\frac{\hat{\mathbf{n}}}{c} \times \frac{\partial \mathbf{A}}{\partial t}\right)/\mu_0 = \left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 \frac{\hat{\mathbf{n}}}{c\mu_0} - \cancel{\left(\frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\hat{\mathbf{n}}}{c}\right)\frac{\partial \mathbf{A}}{\partial t}}/\mu_0$$

Where the last expression follows from the vector triple product rule. Now the expressions from **12** can be substituted and by writing $k\hat{\mathbf{n}} = \mathbf{k}$ and $\omega = ck$ we obtain the answer.