Radiation and Matter

Mathematical Appendix *(a non-examinable addition to the notes)*

Vector Identities

Stokes theorem

$$\int_{area} \nabla_\wedge \mathbf{F} \,.\, \mathrm{d}\mathbf{A} = \oint_{edge} \mathbf{F} \,.\, \mathrm{d}\mathbf{l}$$

Gauss' divergence theorem

$$\int_{volume} \nabla.\ \mathbf{F}\ \mathrm{d}V = \int_{area} \mathbf{F} \,.\, \mathrm{d}\mathbf{A}$$

Identities

$$\nabla_\wedge \nabla \chi = 0, \text{ any } \chi(\mathbf{r})$$

$$\nabla.\ \nabla_\wedge \mathbf{X} = 0, \text{ any } \mathbf{X(r)}$$

$$\nabla_\wedge \nabla_\wedge \mathbf{X} = \nabla\nabla.\mathbf{X} - \nabla^2 \mathbf{X}, \text{ any } \mathbf{X(r)}$$

$$\nabla\ .\ (\mathbf{X}_\wedge \mathbf{Y}) = \mathbf{Y}\ .\ \nabla_\wedge \mathbf{X} - \mathbf{X}\ .\ \nabla_\wedge \mathbf{Y}, \text{ any } \mathbf{X(r), Y(r)}$$

$$\nabla(\mathbf{X}\ .\ \mathbf{Y}) = \mathbf{X}_\wedge \nabla_\wedge \mathbf{Y} + \mathbf{Y}_\wedge \nabla_\wedge \mathbf{X} + \mathbf{X}\ .\ \nabla \mathbf{Y} + \mathbf{Y}\ .\ \nabla \mathbf{X}$$

Vector derivatives of waves (note: $f'(x) \equiv df/dx$)

$(\partial/\partial t)[t - \mathbf{n.r}/c] = 1$ so $\partial f/\partial t = f'$
$(\partial/\partial x)[-\mathbf{n.r}/c] = -n_x/c,$ so $\partial f/\partial x = (-n_x/c)\ f'$ and also for y, z.

Hence

$$\begin{aligned} \nabla f(t - \mathbf{n.r}/c) &= \mathbf{e}_x \partial f/\partial x + \mathbf{e}_y \partial f/\partial y + \mathbf{e}_z \partial f/\partial z \\ &= \mathbf{e}_x\ (-n_x/c)\ f' + \mathbf{e}_y\ (-n_y/c)\ f' + \mathbf{e}_z\ (-n_z/c)\ f' \\ &= -(\mathbf{n}/c)\ f' \\ &= -(\mathbf{n}/c)\ (\partial f/\partial t) \end{aligned}$$

By similar arguments

$$\nabla.\ \mathbf{F}(t - \mathbf{n.r}/c) = \sum_{i=x,y,z} (\partial/\partial x_i) F_i = -(\mathbf{n}/c)\ .\ (\partial \mathbf{F}\ /\partial t)$$

and

$$\begin{aligned} [\nabla_\wedge \mathbf{F}(t - \mathbf{n.r}/c)]_x &= (\partial/\partial y) F_z - (\partial/\partial z) F_y \\ &= (-n_y/c)\ F'_z - (-n_z/c)\ F'_y \\ &= [(-\mathbf{n}/c)_\wedge \mathbf{F}']_x = [(-\mathbf{n}/c)_\wedge (\partial \mathbf{F}\ /\partial t)]_x \end{aligned}$$

and also for y, z, *i.e.*

$$\nabla_\wedge \mathbf{F}(t - \mathbf{n.r}/c) = -(\mathbf{n}/c)_\wedge (\partial \mathbf{F}\ /\partial t)$$

Poynting vector

Manipulate Maxwell's equations to get

$$\mathbf{E}\,.\,\nabla_\wedge\mathbf{B} - \mathbf{B}\,.\,\nabla_\wedge\mathbf{E} = \mu_0\left(\mathbf{j}\,.\,\mathbf{E} + \frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right)\right)$$

The LHS is $-\nabla\,.\,(\mathbf{E}_\wedge\mathbf{B})$. Integrate over a given volume :

$$-\frac{1}{\mu_0}\int\nabla\,.\,(\mathbf{E}_\wedge\mathbf{B})\,dV = \left(\int\mathbf{j}\,.\,\mathbf{E}\,dV + \frac{\partial}{\partial t}\int\left(\frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right)\,dV\right)$$

On the RHS the first integral is the rate at which the current does work, while the second integral is the rate of increase of energy stored in the fields. Therefore the LHS is the energy entering the system. By the divergence (Gauss') theorem, the integral on the LHS is equal to the outward-pointing surface integral of $-\mathbf{E}_\wedge\mathbf{B}/\mu_0$, which implies that the energy flux is $\mathbf{E}_\wedge\mathbf{B}/\mu_0$. (QED - but only generally true in wave zone!)

Solution of $\Box^2\phi(\mathbf{r},t) = -\rho(\mathbf{r},t)/\epsilon_0$

A. First solve $\nabla^2\phi = -\rho(\mathbf{r})/\epsilon_0$:

General result for two scalar fields ϕ,ψ (Green's Theorem) :

$$\int(\phi\nabla\psi - \psi\nabla\phi)\,.\,d\mathbf{A} = \int\nabla\,.\,(\phi\nabla\psi - \psi\nabla\phi)\,dV = \int(\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV$$

where $d\mathbf{A}$ is an element of area.

Choose ϕ, ψ so that each drops off sufficiently rapidly that the area integral is zero when the surface is at infinity. In that case, the left-hand side of these equalities is zero.

Choose in particular $\psi = 1/r$. Note that

$$\nabla^2\left(\frac{1}{r}\right) \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = 0$$

except at $r = 0$.

Check whether there is a spike in $\nabla^2(1/r)$ at $r = 0$ by integrating over volume :

$$\int\nabla^2\left(\frac{1}{r}\right)\,dV = \int\nabla\left(\frac{1}{r}\right)\,.\,d\mathbf{A} = \int\left(-\frac{1}{r^2}\right)\,r^2\,d\Omega = -4\pi.$$

There *is* a spike. Thus

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r})$$

Now insert this in the RHS of Greens's theorem with the surface integral zero :

$$\int\left(\phi[-4\pi\delta(\mathbf{r})] - \frac{1}{r}\nabla^2\phi\right)\,dV = 0,$$

i.e.

$$\phi(0) = -\frac{1}{4\pi}\int\frac{\nabla^2\phi}{r}\,dV = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}'')}{r''}\,dV''.$$

Now shift the origin of coordinates by $\mathbf{r}$ ($\mathbf{r}' = \mathbf{r}'' + \mathbf{r}$) to get

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV'$$

(QED)

B. Now do the time-dependent case : $\nabla^2\phi - c^{-2}\ddot{\phi} = -\rho(\mathbf{r}, t)/\epsilon_0$

First of all, Fourier transform the equation w.r.t. time.

$$\frac{\partial^2}{\partial t^2}(...) \to -\omega^2(...).$$

The equation with Fourier transformed quantities indicated by overbars becomes, on putting $\omega/c = k$,

$$\nabla^2\bar{\phi} + k^2\bar{\phi} = -\bar{\rho}(\mathbf{r}, \omega)/\epsilon_0$$

Instead of $1/r$ use $e^{\pm ikr}/r$ in an analysis similar to case **A**. Check that

$$(\nabla^2 + k^2)\frac{e^{\pm ikr}}{r} = -4\pi\delta(\mathbf{r}).$$

Now note that in Green's theorem, since k is constant, we can replace $\int(\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV$ by

$$\int \Big(\phi(\nabla^2 + k^2)\psi - \psi(\nabla^2 + k^2)\phi\Big)\, dV.$$

Now we we put $\psi = e^{\pm ikr}/r$ (and $\phi \to \bar{\phi}$) to get

$$\bar{\phi}(0, \omega) = -\frac{1}{4\pi}\int \frac{e^{\pm ikr}(\nabla^2 + k^2)\bar{\phi}}{r}\, dV = \frac{1}{4\pi\epsilon_0}\int \frac{e^{\pm ikr}\bar{\rho}(\mathbf{r}'', \omega)}{r''}\, dV''.$$

Now Fourier transform back, and remember that the transform of

$$\bar{\rho}(\mathbf{r}'', \omega)e^{\pm i\omega r/c}$$

is $\rho(\mathbf{r}'', t \pm r/c)$, and then – as before – shift origin to get

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t \pm |\mathbf{r}' - \mathbf{r}|/c)}{|\mathbf{r} - \mathbf{r}'|}\, dV'$$

(QED)

Field energy as vector potential wave amplitudes (section 1.4)

In a large empty cubical box of size L ($V = L^3$) (and in a gauge in which $\nabla.\mathbf{A} = 0$, $\phi = 0$, so that $\mathbf{E} = -\dot{\mathbf{A}}$, $\mathbf{B} = -\nabla_\wedge \mathbf{A}$) the vector potential can be written, by Fourier's theorem, as

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}}\, \mathbf{e}_{\mathbf{k},\alpha} \left(a_{\mathbf{k},\alpha}(t) e^{i\mathbf{k}.\mathbf{r}} \; + \; a^*_{\mathbf{k},\alpha}(t) e^{-i\mathbf{k}.\mathbf{r}} \right),$$

where $\mathbf{k} = (2\pi/L)(n_x, n_y, n_z)$ and the n's are positive or negative integers, and the complex conjugate term has been introduced to ensure that $\mathbf{A}$ is a real function. Since $\mathbf{A}$ obeys the wave equation in empty space, $a_{\mathbf{k},\alpha}(t)$ is proportional to $\exp(-i\omega t)$, $\omega = |\mathbf{k}|c$. The energy in the box is

$$H = \int_{box} \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0)\, dV.$$

The squared fields can be expressed as double sums: for example,

$$\mathbf{E}^2 = -\frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\alpha} \sum_{\mathbf{k}',\alpha'} \sqrt{\omega\omega'} (\mathbf{e}_{\mathbf{k},\alpha}.\mathbf{e}_{\mathbf{k}',\alpha'})(a_{\mathbf{k},\alpha} e^{i\mathbf{k}.\mathbf{r}} - a^*_{\mathbf{k},\alpha} e^{-i\mathbf{k}.\mathbf{r}})(a_{\mathbf{k}',\alpha'} e^{i\mathbf{k}'.\mathbf{r}} - a^*_{\mathbf{k}',\alpha'} e^{-i\mathbf{k}'.\mathbf{r}})$$

and the magnetic term (remember $c^2 = 1/\mu_0\epsilon_0$ and inside the sum $\nabla_\wedge \to \pm i\mathbf{k}_\wedge$) becomes

$$\mathbf{B}^2 = -\frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\alpha} \sum_{\mathbf{k}',\alpha'} \sqrt{\omega\omega'} ((\mathbf{e}_{\mathbf{k},\alpha \wedge} \mathbf{k}).(\mathbf{e}_{\mathbf{k}',\alpha' \wedge} \mathbf{k}'))(a_{\mathbf{k},\alpha} e^{i\mathbf{k}.\mathbf{r}} - a^*_{\mathbf{k},\alpha} e^{-i\mathbf{k}.\mathbf{r}})(a_{\mathbf{k}',\alpha'} e^{i\mathbf{k}'.\mathbf{r}} - a^*_{\mathbf{k}',\alpha'} e^{-i\mathbf{k}'.\mathbf{r}})$$

All the terms in the sums representing the $\mathbf{E}$ and $\mathbf{B}$ fields contain a factor $e^{\pm i\mathbf{k}.\mathbf{r}}$, the only factor depending on $\mathbf{r}$. The integral involves these terms only, and since the fields are squared the terms appear only as products of pairs. The integral can be vastly simplified by noting that the integral of such pairs is

$$\int_{box} e^{i(\mathbf{k}'+\mathbf{k}'').\mathbf{r}}\, dV = V \delta_{\mathbf{k}',-\mathbf{k}''}$$

where $\delta_{\mathbf{k}',-\mathbf{k}''}$ is unity if $\mathbf{k}' = -\mathbf{k}''$, and zero otherwise.

Thus in the energy integral it is clear that *all* the terms will vanish on integration except those that arise by multiplying terms of a given $\mathbf{k}$ and their complex conjugates, plus terms (e.g. $a_{\mathbf{k},\alpha} a^*_{-\mathbf{k},\alpha}$) that cancel between the E^2 and the B^2 sums. The vector dot products in the sums ensure that non-zero terms have $\alpha' = \alpha$. The upshot is

$$H = \frac{1}{2} \sum_{\mathbf{k},\alpha} \hbar\omega (a_{\mathbf{k},\alpha} a^*_{\mathbf{k},\alpha} + a^*_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha}).$$

Gauge invariance and electromagnetism (without Lagrangians)

$$i\hbar\left(\frac{\partial}{\partial t}+f\right)\psi = H(\nabla+\mathbf{F})\psi$$

with

$$f \to f' = f - i\dot{\alpha}, \qquad \mathbf{F} \to \mathbf{F}' = \mathbf{F} - i\nabla\alpha$$

is invariant to the gauge transformation $\psi \to e^{i\alpha}\psi$ (NB *local* gauge invariance is implied by relativity, since the phase ('gauge') change α cannot be propagated at infinite speed, and so must be different in different places at different moments). The extra necesssary fields $f, \mathbf{F}$ turn out to be the EM potentials within a factor. For the fields $\mathbf{M} = \nabla_\wedge \mathbf{F}$ and $\mathbf{N} = \dot{\mathbf{F}} - \nabla f$ are gauge invariant, and

$$\nabla_\wedge \mathbf{M} = 0, \qquad \dot{\mathbf{M}} = \nabla_\wedge \mathbf{N}$$

are the first two Maxwell equations. To check that this really is so, and to set the constant factors,we use a semi-classical argument to derive the Lorentz force (only really done properly in Lagrangian formalism).

The Schrödinger equation of a free particle (with these gauge fields implied) is

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\mathbf{p}+\hbar\mathbf{F}/i)^2 - i\hbar f$$

and the equation can then be written (assuming $(\hbar\mathbf{F})^2$ is small enough to ignore)

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\mathbf{p}^2 + V$$

where V, the effective potential energy, is $i\hbar(\mathbf{v}.\mathbf{F}+f)$. Then the classical equation of motion would be

$$\frac{d\mathbf{p}}{dt} = -\nabla V.$$

Note that $\mathbf{v}$ is the trajectory (yet to be found) of the particle, and so

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}.\nabla\mathbf{A}$$

and using the identity for $\nabla(\mathbf{v}.\mathbf{A})$, – don't differentiate $\mathbf{v}$, as it is a trajectory, not a field variable! – we get

$$\frac{d}{dt}(\mathbf{p} - q\{i\hbar/q\}\mathbf{F}) = q(\{-i\hbar/q\}\mathbf{N} + \mathbf{v}_\wedge\{i\hbar/q\}\mathbf{M}).$$

If, putting $i\hbar/q = K$, we write $\mathbf{B} = K\mathbf{M}$, $\mathbf{E} = -K\mathbf{N}$, $\mathbf{A} = K\mathbf{F}$, $\phi = -Kf$, then the equation becomes

$$\frac{d}{dt}(\mathbf{p} - q\mathbf{A}) = q(\mathbf{E} + \mathbf{v}_\wedge\mathbf{B}),$$

which is the Lorentz equation, and in addition we have learned that the momentum in an EM field is not $\mathbf{p}$, but $\mathbf{p} - q\mathbf{A}$. Finally, the gauge invariant S. equation is (with $\mathbf{p} \equiv (\hbar/i)\nabla$)

$$\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\psi = H(\mathbf{p} - q\mathbf{A})\,\psi \simeq \frac{(\mathbf{p}-q\mathbf{A})^2}{2m}\psi.$$

Maxwell averaged cross section

The cross section for a beam of particles of type 1, travelling with velocity v, colliding with a static particle of type 2 ('colliding with' can mean reacting with or being deflected by) is $\sigma = \sigma(v)$. In a gas at temperature T consisting of n_1 m^{-3} particles of type 1 and n_2 m^{-3} particles of type 2, the rate of collision is

$$n_1 n_2 \langle \sigma v \rangle \quad \mathrm{m}^{-3}\ \mathrm{s}^{-1}$$

This is proven in two steps. First, choose a coordinate frame moving with velocity $\mathbf{v}_2$. Consider the flux of particles 1 on a particle 2 which is at rest in this frame. This will be given by σ times the flux of particles 1, *i.e.* n_1 times the relative velocity $v = |\mathbf{v}_1 - \mathbf{v}_2|$, integrated over the Maxwell velocity distribution of particles 1. This distribution is

$$f(\mathbf{v}_1; m_1)\, d^3\mathbf{v}_1 = \left(\frac{m_1}{2\pi kT}\right)^{3/2} e^{-m_1 v_1^2/2kT}\, d^3\mathbf{v}_1.$$

The result is

$$n_1 \int_{\mathbf{v}_1} \sigma v f(\mathbf{v}_1; m_1)\, d^3\mathbf{v}_1.$$

Now, in unit volume and in a small velocity range (nearly at rest in the moving frame) there are $n_2 f(\mathbf{v}_2; m_2)\, d^3\mathbf{v}_2$ particles 2. Thus the number of collisions m^{-3} $\mathrm{s}^{-1}$of particles 1 with particles 2 is

$$n_1 n_2 \int_{\mathbf{v}_1} \int_{\mathbf{v}_2} \sigma v f(\mathbf{v}_1; m_1)\, f(\mathbf{v}_2; m_2)\, d^3\mathbf{v}_1\, d^3\mathbf{v}_2.$$

Now change variables from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{V} = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/(m_1 + m_2)$ (*i.e.* to relative and centre-of-mass velocities). Note that $m_1 m_2 = \mu M$ where $M = m_1 + m_2$ and *reduced mass* $\mu = m_1 m_2 / M$, and $m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2 = \mu\mathbf{v}^2 + M\mathbf{V}^2$. Also (the Jacobian has magnitude unity!) $d^3\mathbf{v}_1\, d^3\mathbf{v}_2 = d^3\mathbf{v}\, d^3\mathbf{V}$. Substituting in the integrals, they become

$$n_1 n_2 \int_{\mathbf{v}} \int_{\mathbf{V}} \sigma v f(\mathbf{v}; \mu)\, f(\mathbf{V}; M)\, d^3\mathbf{v}\, d^3\mathbf{V}.$$

The integrals separate, that over $\mathbf{V}$ integrating to unity (normalized velocity distribution), while that over $\mathbf{v}$ is the Maxwell average of σv for a particle of mass μ. this average is the one-dimensional integral (express $d^3\mathbf{v}$ in polar coordinates $v^2 dv\, d\Omega$):

$$\langle \sigma v \rangle = 4\pi \left(\frac{m_1}{2\pi kT}\right)^{3/2} \int_0^\infty \sigma v^3 e^{-\mu v^2/2kT}\, dv.$$

[Note:
1. the collision rate for a binary process – 2 particles – is proportional to density *squared* ($n_1 n_2 = x_1 x_2 n^2$ where x_1, x_2 are abundances). You can guess that the triple collision rate will be proportional to density *cubed.*

2. The coefficient $\langle \sigma v \rangle$ is a function of temperature. If $\sigma v \propto E^\alpha \propto v^{2\alpha}$ then inspection of the integral shows that $\langle \sigma v \rangle \propto T^\alpha$.]

Spin

Rules of angular momentum:

$$[j_x, j_y] = i\hbar j_z, \quad \text{(cyclic)}$$

hence

$$[\mathbf{j}^2, j_z] = 0 \quad \text{(for any one cpt.)}$$

and the complete set of states are specified by the eigenvalues of these two operators. Invent $j_\pm = j_x \pm i j_y$, and note $(j_+)^\dagger = j_-$ and $\mathbf{j}^2 = j_\pm j_\mp + j_z^2 \mp j_z$. Hence it is found that there are 2j+1 states, with z-component eigenvalues $m = -j, -(j-1), \ldots, (j-1), j$ in units of $\hbar$; the eigenvalue of squared AM is $j(j+1)$ in these units; and

$$j_\pm |j,\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j,\ m \pm 1\rangle.$$

The wave functions corresponding to the states with $2j$ even are $\langle x|j, m\rangle = Y_{jm}(\theta, \phi)$, the spherical harmonics. for $2j$ odd there are no such wave functions (we use 2-component *spinors*).

Adding two AM operators (a system with two angular momenta), $\mathbf{J} = \mathbf{j}(1) + \mathbf{j}(2)$, gives complete sets of operators $\mathbf{j}(1)^2$, $\mathbf{j}(2)^2$, $\mathbf{J}^2$, J_z, or alternatively $\mathbf{j}(1)^2$, $\mathbf{j}(2)^2$, $j(1)_z$, $j(2)_z$ (check that in each group the operators commute). Since each is complete, the eigenstates of one group must be linear combinations of the eigenstates in the other. The coefficients are called *Clebsch-Gordan coefficients*. They can be found using the ladder operators $j_\pm$.

We apply this to two spin $\frac{1}{2}$ particles. The spin operator has two eigenvalues, $\pm 1/2$, and the two eigenstates can be written $|\pm\rangle$. The commuting operators (neglect the $\mathbf{s}(1)^2$, $\mathbf{s}(2)^2$ operators, since their eigenvalues are always the same, 3/4) are $s_z(1)$, $s_z(2)$ or $\mathbf{s}^2$, s_z, where $\mathbf{s} = \mathbf{s}(1) + \mathbf{s}(2)$. The corresponding eigenstates are then either $|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle$ or alternatively $|S, M_S\rangle$. We relate these alternative descriptions by noting that the $S = 1$, $M_S = 1$ state must be both spins 'up', *i.e.* $|1, 1\rangle = |+\rangle|+\rangle$.

To get the $|1,\ 0\rangle$ state, apply $S_- = s_-(1) + s_-(2)$ to the above equation and apply the formula for the action of $j_\pm$ (using S_- on the left and $s_-(1) + s_-(2)$ on the right):

$$|1,\ 0\rangle = \frac{1}{\sqrt{2}} \left(|+-\rangle + |-+\rangle\right),$$

and again to get $|1,\ -1\rangle$:

$$|1,\ -1\rangle = |--\rangle.$$

The remaining state, $|0, 0\rangle$, must be orthogonal to all of these, and in particular to $|1, 0\rangle$, from which we infer

$$|0,\ 0\rangle = \frac{1}{\sqrt{2}} \left(|+-\rangle - |-+\rangle\right)$$

(we could have derived this also by demanding $S_\pm|0,\ 0\rangle = 0$).

Matrix elements of $\mathbf{s}_{\text{electron}}$ between $S = 1, M_S$ and $S = 0, M_S = 0$

The method applies to the spins of the electron and nucleus in a hydrogen atom. The spins combine as above, but transitions are caused by an operator on the electron state only (because the Bohr magneton for baryons is so small). The squared matrix element required is $|\langle Y|\mathbf{s}|X\rangle|^2 = |\mathbf{s}_{YX}|^2$ given by

$$|\mathbf{s}_{YX}|^2 = |(s_x)_{YX}|^2 + |(s_y)_{YX}|^2 + |(s_z)_{YX}|^2 = \left(|(s_+)_{YX}|^2 + |(s_-)_{YX}|^2\right)/2 \ + |(s_z)_{YX}|^2.$$

Take $|XM\rangle$ to be one of the $S = 1$ states, and $|Y\rangle$ to be the $S = 0,\ M = 0$ state, also expressible as

$$|Y\rangle = |0,\ 0\rangle = \frac{1}{2}\left(|+\rangle_e|-\rangle_p - |-\rangle_e|+\rangle_p\right)$$

The $\mathbf{s}$ operator is the electron spin operator, and doesn't affect the proton spin. Check that $s_+|-\rangle_e = |+\rangle_e,\ \ s_-|+\rangle_e = |-\rangle_e,\ \ s_z|\pm\rangle_e = \pm(1/2)|\pm\rangle_e$ (and remember in what follows that the first state vector in the product is the electron one).

For $|XM\rangle$ representing the states with $S = 1$ and $M_S = 1, 0, -1$ we get

M	$\|XM\rangle$	$(s_+)\|XM\rangle$	$(s_-)\|XM\rangle$	$(s_z)\|XM\rangle$	$(s_+)_{YX}$	$(s_-)_{YX}$	$(s_z)_{YX}$
1	$\|+\rangle\|+\rangle$	0	$\|-\rangle\|+\rangle$	$\frac{1}{2}\|+\rangle\|+\rangle$	0	$-\frac{1}{\sqrt{2}}$	0
0	$\frac{1}{\sqrt{2}}(\|+\rangle\|-\rangle + \|-\rangle\|+\rangle)$	$\frac{1}{\sqrt{2}}\|+\rangle\|+\rangle$	$\frac{1}{\sqrt{2}}\|-\rangle\|-\rangle$	$\frac{1}{2\sqrt{2}}(\|+\rangle\|-\rangle - \|-\rangle\|+\rangle)$	0	0	$\frac{1}{2}$
−1	$\|-\rangle\|-\rangle$	$\|+\rangle\|-\rangle$	0	$-\frac{1}{2}\|-\rangle\|-\rangle$	$\frac{1}{\sqrt{2}}$	0	0

The result $|\mathbf{s}_{YX}|^2$ is then $1/4$, the same for each value of M, as it should be for an angle-averaged quantity.

This value, in physical units then gives the term for magnetic dipole radiation

$$|2\mathbf{s}_{YX}|^2 = \hbar^2.$$