Quantum Mechanics 3   2001/2002

Solution set 6

(1) This question is easier than it looks, since the main results of importance are given, and don't need to be proved. This means that the explicit expression for the operator a is a red herring: it is never needed.

For the commutator $[a, H]$, write it in full in terms of a and $a^\dagger$:

$$aH - Ha = \hbar\omega(a^\dagger aa + a/2) - \hbar\omega(aa^\dagger a + a/2) = \hbar\omega[a^\dagger, a]\, a.$$

Using $[a^\dagger, a] = -1$, this is $-\hbar\omega a$. The commutator $[a^\dagger, H]$ is obtained similarly.

Now consider an eigenstate $H\psi = E\psi$ (actually, there will be a set ψ_i with energies E_i), and think about $a^\dagger\psi$: is it possible that this is another of the energy eigenstates? If so, we would have

$$H(a^\dagger\psi) = \beta(a^\dagger\psi),$$

where the eigenvalue β would be the new energy. Now, $Ha^\dagger\psi = [H, a^\dagger]\psi + a^\dagger H\psi$. Using $H\psi = E\psi$ and the negative of the above commutator, this is of the required form, and $\beta = E + \hbar\omega$. Thus, $a^\dagger$ raises the state ψ. The proof that a lowers it is very similar.

Now, the energy eigenvalue must be > 0. We can either argue physically: the potential $V = Kx^2$ is positive, and the kinetic energy must be positive also, so negative energy makes no sense (it would be OK if the potential was negative, like the electrostatic potential). If this seems too classical an argument, consider an eigenstate $H\psi = E\psi$, for which $E = \langle\psi|H|\psi\rangle$. Putting in the definition of H, this says $E = \hbar\omega(\langle\psi|a^\dagger a|\psi\rangle + 1/2)$. But $\langle\psi|a^\dagger a|\psi\rangle = \langle a\psi|a\psi\rangle$ (because operators act to the left when you conjugate them). This matrix element is just $\int |a\psi|^2$, so is > 0.

So, there must exist a ground state ψ_g that can't be lowered, in which case $a\psi_g = 0$. The trick then is to say that therefore also $\hbar\omega a^\dagger a\psi_g = 0$. The operator is $H - \hbar\omega/2$, so $E_g = \hbar\omega/2$. The other energy levels are obtained from ψ_g by repeated application of $a^\dagger$; hence $E = (n + 1/2)\hbar\omega$.

(2)

(a) The definition of the ground state, $u_0(x)$, is that it cannot be lowered. Therefore, trying to lower it gives zero probability for finding the particle in such a lowered state – i.e. zero wavefunction. Therefore $au_0 = 0$. Unlike the Schrödinger equation, this is a first-order differential equation:

$$\left(\alpha x + \alpha^{-1}\frac{\partial}{\partial x}\right) u_0 = 0.$$

(putting in the definition of p). This is simply integrated:

$$\frac{du_0}{u_0} = -\alpha^2 x\, dx \quad \Rightarrow \quad u_0 = A\exp(-\alpha^2 x^2/2),$$

putting in the boundary condition that $u_0(\infty) = 0$ for normalization.

To normalize the wavefunction, we need

$$|A|^2 \int_{-\infty}^{\infty} \exp(-\alpha^2 x^2)\, dx = 1$$

Now, $\int \exp(-y^2)\, dy = \sqrt{\pi}$ [to prove this, square the integral, and change variables from (x, y) to polars (r, θ)]. Therefore

$$|A| = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$

(b) The state $u_1(x)$ comes from raising u_0 once: $u_1(x) \propto a^\dagger u_0(x)$ (not equal: the normalization is affected). In other words:

$$u_1(x) \propto \left(\alpha x - \alpha^{-1}\frac{\partial}{\partial x}\right) u_0 = 2\alpha x\exp(-\alpha^2 x^2/2) = Bx\exp(-\alpha^2 x^2/2),$$

where B is a normalization constant.

(c)

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\alpha^2 x^2}{2}\hbar\omega.$$

We therefore need to differentiate u_0 twice, which gives $u_0'' = -\alpha^2 u_0 + \alpha^4 x^2 u_0$. This cancels the quadratic term, giving

$$Hu_0 = \frac{\alpha^2\hbar^2}{2m} u_0.$$

Hence the energy eigenvalue is $E = \alpha^2\hbar^2/2m = \hbar\omega/2$, as required.