# Quantum Mechanics 3 2001/2002

## Solution set 5

(1)

(a) $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$, so

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right),$$

and the same for other components, cyclically permuting $x, y, z$. The first commutator is just hard work:

$$\begin{aligned} L_x L_y &= -\hbar^2\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ &= -\hbar^2\left(y\frac{\partial}{\partial z} + yz\frac{\partial^2}{\partial z\partial x} - yx\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial y\partial x} + zx\frac{\partial^2}{\partial y\partial z}\right) \end{aligned}$$

Now do the same for $L_y L_x$ and take the difference.

(b) We need to consider $L_z L_\pm$, which is $L_\pm L_z - [L_\pm, L_z]$. The required commutator is readily proved from the basic commutators: $[L_\pm, L_z] = \mp\hbar L_\pm$. Thus, $L_z L_\pm|\ell, m\rangle = (L_\pm L_z \pm \hbar L_\pm)|\ell, m\rangle$. Since $|\ell, m\rangle$ is an eigenstate of $L_z$, this gives $L_z L_\pm|\ell, m\rangle = (m \pm 1)\hbar L_\pm|\ell, m\rangle$, so the state $L_\pm|\ell, m\rangle$ is also an eigenstate of $L - z$, but its eigenvalue is $(m \pm 1)\hbar$ – this is what we mean by raising or lowering.

(c) $L_x = (L_+ + L_-)/2$. If $\psi$ is the $m$ eigenstate of $L_z$, $L_x$ produces a mixture of the $m - 1$ and $m + 1$ states, which are orthogonal to the $m$ state. Hence, $\langle L_x\rangle = 0$. Similarly,

$$L_x^2 = \left(\frac{L_+ + L_-}{2}\right)^2 = (L_+^2 + L_-^2 + L_+L_- + L_-L_+)/4.$$

The expectation of the first two terms vanishes, leaving $\langle L_x^2\rangle = \langle L_+L_- + L_-L_+\rangle/4$. Now, $L_+L_- + L_-L_+ = 2(L_x^2 + L_y^2) = 2(L^2 - L_z^2)$, from the definition of $L_\pm$. Hence, $\langle L_x^2\rangle = 2\langle(L^2 - L_z^2)\rangle/4 = \hbar^{(}\ell(\ell + 1) - m^2)/2$. The last step can also be argued by symmetry: $\langle(L^2 - L_z^2)\rangle = \langle L_x^2\rangle + \langle L_y^2\rangle$, but we expect $\langle L_x^2\rangle = \langle L_y^2\rangle$.

(d) $\langle(\delta A)^2\rangle$ means $\langle(A-\langle A\rangle)^2\rangle$, which is $\langle A^2\rangle - \langle A\rangle^2$. Here, $\langle(\delta L_x)^2\rangle = \langle L_x^2\rangle$, and similarly for $L_y$ (from the previous part). Again using the previous part, we get

$$\langle(\delta L_x)^2\rangle\langle(\delta L_y)^2\rangle = \frac{\hbar^4}{4}[\ell(\ell+1)-m^2]^2.$$

Since the maximum values of $m$ is $\ell$, this is $\langle(\delta L_x)^2\rangle\langle(\delta L_y)^2\rangle \geq \hbar^4\ell^2/4$.

Now consider the general uncertainty relation:

$$\langle(\delta A)^2\rangle\langle(\delta B)^2\rangle \geq -\langle[A,B]\rangle^2/4.$$

we have $\langle[L_x,L_y]\rangle = i\hbar\langle L_z\rangle = im\hbar^2$, so the rhs of the uncertainty relation is $\hbar^4 m^2/4$. However, we already proved that the lhs was $\geq \hbar^2\ell^4/4$, and $\ell \geq m$, so this is consistent with the uncertainty relation (which becomes an equality when $m=\ell$).

(2) Using $L^2 = L_x^2 + L_y^2 + L_z^2$, write out the commutator with total angular momentum squared:

$$\begin{aligned}[L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\ &= [L_y^2, L_x] + [L_z^2, L_x] \\ &= L_yL_yL_x - L_xL_yL_y + L_zL_zL_x - L_xL_zL_z\end{aligned}$$

Now use the $[L_x, L_y] = i\hbar L_z$ commutator to convert the triple terms to 'sandwiches' plus pairs: $L_yL_yL_x = L_yL_xL_y - i\hbar L_yL_z$ etc. The sandwiches cancel in pairs, and the four double terms combine to give zero. Once you've proved it for $L_x$, the other two follow just by symmetry – the choice of $x$ axis is arbitrary.

(3) This mainly needs stamina in changing variables to spherical polars:

$$\begin{aligned}x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta.\end{aligned}$$

Start with the chain rule for partial derivatives: the partial derivative with respect to $\phi$ is

$$\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

which gives $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$.

The same reasoning for $\theta$ gives

$$\frac{1}{r}\frac{\partial}{\partial\theta} = \cos\theta\cos\phi\frac{\partial}{\partial x} + \cos\theta\sin\phi\frac{\partial}{\partial y} - \sin\theta\frac{\partial}{\partial z}.$$

Together with the relation for $\frac{\partial}{\partial\phi}$, we can eliminate $\frac{\partial}{\partial x}$, which is what we need in order to involve $L_x = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$. The following combination can be checked to work:

$$L_x/i\hbar = \quad \sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}.$$

The same approach gives the equation for $L_y$:

$$L_y/i\hbar = -\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}.$$

(remember $\cot = 1/\tan$).

The expression for $L^2$ involves keeping track of the cross terms in which the angular derivatives in $L_x$ and $L_y$ differentiate the operators themselves. This gives an extra $\cot\theta\frac{\partial}{\partial\theta}$ term, so

$$\frac{L^2}{(i\hbar)^2} = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta}.$$

Finally, $\nabla^2$ in spherical polars is

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2},$$

so we can see that the angular terms are proportional to $L^2$.