Quantum Mechanics 3   2001/2002

Solution set 3

(1) This is mainly bookwork:

(a) Equation (fifi) in the notes defines $O^\dagger$.

(b) A Hermitian operator has $O^\dagger = O$. Answers in terms of matrices would get marks – but the question wanted you to use the general operator definition.

(c) Again, this is in the notes (equations 82 & 83). The key step is to be able to say that the 'boundary term' $[\psi_1^*\psi_2] = 0$. We haven't been explicit about the range of integration, but it is always all space, so we need $[\psi_1^*\psi_2]^{\infty}_{-}\infty$. The boundary term vanishes because ψ_1 and ψ_2 must vanish at infinity. To see this, assume the opposite: that they tend to some constant. $\int |\psi|^2 dx$ would then diverge and the wavefunction wouldn't be normalizable. So, $(d/dx)^\dagger = -(d/dx)$. Note that this minus sign gets lost if you do the same exercise for $i(d/dx)$. Thus, the momentum operator is Hermitian.

(d) This is a bit tedious, but you just need to keep using the definition of the conjugate:

$$\left(\int \psi_1^*(AB)\psi_2 \; dV\right)^* = \int \psi_2^*(AB)^\dagger\psi_1 \; dV.$$

Now, recognize that $B\psi_2$ is just some other function, which we can call ψ_3. Using the definition,

$$\left(\int \psi_1^* A\psi_3 \; dV\right)^* = \int \psi_3^* A^\dagger\psi_1 \; dV.$$

Now do the same thing again, with $A^\dagger\psi_1 = \psi_4$, and write

$$\int \psi_3^*\psi_4 \; dV = \left(\int \psi_4^*\psi_3 \; dV\right)^*$$

(because ψ_3^* and ψ_4 commute, and taking a $*$ outside the whole integral). Re-using $\psi_3 = B\psi_2$ brings in $B^\dagger$, and reordering terms brings $B^\dagger A^\dagger$ together in the middle.

(2) Sketch ψ: the question says $\psi = A$ (some constant) in the left half of the well. It also says the particle is placed in the left side, which must mean $\psi = 0$ in the right half. Normalization requires $\int_0^{a/2} |A|^2 dx = 1$, or $|A| = \sqrt{2/a}$. You may worry that ψ doesn't satisfy the boundary conditions, which require $\psi(0) = 0$. This just means ψ is in the form of a rectangular 'top hat' that falls to zero at $x = 0$ and $x = a/2$.

Now expand this funny state in the states of well-defined energy: $\psi(x) = \sum_i a_i u_i(x)$. The probability of being in the ground state ($i = 1$) is $|a_1|^2$ (see discussion at the top of p24 of the notes), so we need a_1. The coefficients are extracted using orthonormality of the u_i, by multiplying the basic expansion by U_k^* and integrating:

$$u_k^* \psi = \sum_i a_i u_k^* u_i \quad \Rightarrow \quad \int u_k^* \psi \; dV = \sum_i a_i \int u_k^* u_i \; dV = a_k.$$

The last step arises through orthonormality: $\int u_k^* u_i \; dV$ is unity when $i = k$ and zero otherwise.

Putting in the expressions for u_i and ψ, this gives $a_i = (2/ak)[1 - \cos(ka/2)]$, where $ka/2 = n\pi/2$. For $n = 1$, this is just $a_1 = 2/\pi$, so the required probability is $4/\pi^2 \simeq 0.4053$.

(3) This question actually requires very little work, beyond a familiarity with the solution of the quantum harmonic oscillator. For $x > 0$, the given potential is identical to that of the harmonic oscillator. Therefore, the general solution of the Schrödinger equation for $x > 0$ is the same as that for the harmonic oscillator. In the notes, we showed that this was $\psi(x) = F(z)\exp(-z^2/2)$, where $z = x/(\hbar/m\omega)^{1/2}$. We looked for a power-series solution for the unknown function $F(z)$ and showed that it must be a finite polynomial, otherwise the wave function would diverge as $x \to \infty$. The recurrence relation for the coefficients of the polynomial showed that the functions $F(z)$ are either odd or even, with energy eigenvalues $E_n = (n + 1/2)\hbar\omega$, and that $F(z)$ is even if n is even and odd if n is odd.

All of this reasoning applies identically to the $x > 0$ region of the potential given in the question. However, because the potential is infinite for $x < 0$, there is an extra boundary condition to satisfy: $\psi(x) = 0$ at $x = 0$. This is true for only half of the allowed states for the harmonic oscillator: the odd ones. Therefore, the allowed energy levels are $E_n = (n + 1/2)\hbar\omega$, with $n = 1, 3, 5, \ldots$ Another way of writing this is to put $n = 1 + 2m$: $E_m = (2m + 3/2)\hbar\omega$, with $m = 0, 1, 2, 3, \ldots$