Quantum Mechanics 3 2001/2002

Solution set 2

(1)

(a) Stationary state means that the wavefunction factorizes: $\psi(x,t) = u(x)\exp(-iEt/\hbar)$. This is the simple time dependence for a state of definite energy. Whenever you see just $u(x)$, think of this as $\psi(x,t=0)$ – so that there will always be an implicit factor $\exp(-iEt/\hbar)$ for the time dependence.

The full Schrödinger equation is $-(\hbar^2/2m)\nabla^2\psi + V\psi = i\hbar\partial\psi/\partial t$. Putting in the stationary state expression converts the rhs to just $E\psi$, giving the time-independent Schrödinger equation. Notice that the full Schrödinger equation allows superposition of solutions, but the time-independent form does not, because each state has a different energy: each state solves a different time-independent equation.

(b) Therefore, when you see a superposition like $[u_1(x) + u_2(x)]/\sqrt{2}$, this must be the $t=0$ form of the time-dependent superposition $[\psi_1(x,t)+\psi_2(x,t)]/\sqrt{2}$. At a later time, the full expression for ψ is

$$\psi = [u_1(x)\exp(-iE_1t/\hbar) + u_2(x)\exp(-iE_2t/\hbar)]/\sqrt{2},$$

and it is crucial that the two energies are different. The complex conjugate is

$$\psi^* = [u_1^*(x)\exp(+iE_1t/\hbar) + u_2^*(x)\exp(+iE_2t/\hbar)]/\sqrt{2},$$

so

$$|\psi|^2 = |u_1|^2/2 + |u_2|^2/2 + (u_1^*u_2/2)\exp[i(E_1-E_2)t/\hbar] + (u_1u_2^*/2)\exp[-i(E_1-E_2)t/\hbar].$$

If $u_1^*u_2 = |u_1u_2|\exp(i\phi)$, then

$$|\psi|^2 = |u_1|^2/2 + |u_2|^2/2 + |u_1u_2|\cos[(E_1-E_2)t/\hbar + \phi].$$

In other words, the probability density oscillates back and forth. Quantum interference changes the probability density as a function of time.

(2)

(a) The Schrödinger equation for this case is

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + m\omega^2 x^2/2\right)\psi - E\psi = 0$$

(b) If $\psi = Ae^{-\alpha x^2}$, then the first 2 derivatives are

$$\begin{aligned}\psi' &= -2\alpha x A e^{-\alpha x^2} \\ \psi'' &= -2\alpha e^{-\alpha x^2} + 4\alpha^2 x^2 A e^{-\alpha x^2}.\end{aligned}$$

The resulting Schrödinger equation must be true for all x, so the total x^2 coefficient in the equation must be zero: $-2\hbar^2\alpha^2/m + m\omega^2/2 = 0$, so $\alpha = m\omega/2\hbar$. Similarly, the constant term in the equation must vanish: $\hbar^2\alpha/m - E = 0$, so $E = \hbar\omega/2$ – which proves the $n = 0$ case of $E = (n + 1/2)\hbar\omega$.

(3)

(a) Parity: $\psi(-x) = \pm\psi(x)$ (can always assume definite parity for a symmetric potential). Between the delta-potentials, $V = 0$, so the allowed wavefunctions are $\psi \propto \exp(\pm\beta x)$, where $\beta^2 = -2mE/\hbar^2$. here, we want a bound state, so E is negative and β is a real number. The positive parity solution therefore is

$$\psi = \begin{cases} Ae^{\beta x} & x < -a \\ B(e^{\beta x} + e^{-\beta x}) & -a < x < a \\ Ae^{-\beta x} & x > a \end{cases}$$

and for negative parity

$$\psi = \begin{cases} Ae^{\beta x} & x < -a \\ B(e^{\beta x} - e^{-\beta x}) & -a < x < a \\ -Ae^{-\beta x} & x > a \end{cases}$$

(using the only 2 allowed waves to make combinations with the right symmetry). The waves between the spikes are obviously cosh or sinh functions.

(b) As in the notes, integrate the Schrödinger equation across a delta function of weight A to get $\Delta\psi' = (2mA/\hbar^2)\psi$. If you're worried about internal structure in the well, remember k inside is $\propto \sqrt{V}$ when V is large. The phase change across the well is KL, which is $\propto \sqrt{VL}\sqrt{L}$. If we keep VL fixed and let $L \to 0$, this tend to zero. We can treat ψ as constant inside the well.

(c) The boundary condition in (b) gives the same info at $x = \pm a$ by symmetry. Matching ψ and writing down the discontinuity in ψ' gives 2 equations which can be divided to eliminate A and B. We get

$$\frac{2m|\alpha|}{\hbar^2} - \beta = \begin{cases} \beta\tanh\beta a & \text{(even parity)} \\ \beta/\tanh\beta a & \text{(odd parity)} \end{cases}$$

Sketch the lhs and rhs of these expressions against β. There is always an intersection for the even case, but not for the odd case if $1/a > 2m|\alpha|/\hbar^2$.