Quantum Mechanics 3 2001/2002

Problem set 8

(1) [June 1998 degree exam, slightly modified] Two identical spin-zero bosons are placed in a one-dimensional square potential well with infinitely high walls: $V = 0$ for $0 < x < L$, otherwise $V = \infty$. The normalized single-particle energy eigenstates are $u_n = (2/L)^{1/2} \sin(n\pi x/L)$.

(a) Find the wavefunctions and energies for the ground state and the first two excited states of the system.

(b) Suppose that the two bosons interact with each other through the perturbative potential

$$H'(x_1, x_2) = -LV_0 \delta(x_1 - x_2).$$

Compute the first-order perturbation to the ground-state energy of the system.

(2) A one-dimensional potential well with infinitely high walls runs from $x = 0$ to $x = L$; the normalized energy eigenstates are $u_n(x) = (2/L)^{1/2} \sin(n\pi x/L)$. Two identical non-interacting spin-1/2 particles are placed in the well.

(a) What are the allowed values of the total spin angular momentum quantum number, j? How many possible values are there for the z component of total angular momentum?

(b) If single-particle spin eigenstates are denoted by $|\uparrow\rangle \equiv u$ and $|\downarrow\rangle \equiv d$, construct two-particle spin states that are either symmetric or antisymmetric. How many states of each type are there?

(c) Show that the $j = 1, m = 1$ state must be symmetric. What is the symmetry of the $j = 0$ state?

(d) What is the ground-state energy of the two-particle system, and how does it depend on the overall spin state?

[PTO]

(3) [May 1996 degree exam, with one extra part] Write down the commutation relations between the operators J_x, J_y and J_z which represent the Cartesian components of angular momentum.

(a) Show that the operators $J_\pm = J_x \pm iJ_y$ act as raising and lowering operators for the the z component of angular momentum, by first calculating the commutator $[J_z, J_\pm]$.

(b) State the allowed values of the total spin angular momentum for a system of three electrons.

(c) The 'coupled basis' state $|s = 3/2, m_s = 3/2\rangle$ (eigenstate of total spin) is also a state of the 'uncoupled basis', which may be denoted by $|\uparrow\uparrow\uparrow\rangle$. By an application of lowering operators, show that

$$|s = 3/2, m_s = 1/2\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right).$$

(d) Use the same approach to prove that, for the addition of two spins, the singlet state is antisymmetric: $\chi_{0,0} = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$ (use the fact that $J_-\chi_{0,0} = 0$).

You may assume the general result $J_\pm|j, m\rangle = [j(j+1) - m(m \pm 1)]^{1/2}\hbar|j, m \pm 1\rangle$.