Quantum Mechanics 3 2001/2002

Problem set 7

(1) [June 1998 degree exam] A beam of spin-1/2 particles travelling in the y direction, is sent through a Stern-Gerlach apparatus, which is aligned in the z direction, and which divides the incident beam into two beams with $m = \pm 1/2$. The $m = 1/2$ beam is allowed to impinge on a second Stern-Gerlach apparatus aligned along the direction $\mathbf{e} = (\sin\theta)\hat{\mathbf{x}} + (\cos\theta)\hat{\mathbf{z}}$.

(a) Evaluate $S = (\hbar/2)\boldsymbol{\sigma} \cdot \mathbf{e}$, where $\boldsymbol{\sigma}$ is represented by the Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Calculate the eigenvalues of S.

(b) Calculate the normalized eigenvectors of S.

(c) Calculate the intensities of the two beams which emerge from the second Stern-Gerlach apparatus.

(2) [September 2000 resit paper] Ignoring spin, a given electronic level in the Hydrogen atom is characterized by three quantum numbers: (n, ℓ, m). Give the allowed values of ℓ and m for a given value of n, and hence calculate the degeneracy as a function of n – you may assume that $\sum_{i=1}^{N} i = N(N+1)/2$.

How is the degeneracy altered by the existence of spin?

(3) The time-independent Schrödinger equation for a spherically symmetric potential $V(r)$ is

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}R\right] = (E - V)R,$$

where $\psi = R(r)Y_\ell^m(\theta, \phi)$, so that the particle is in an eigenstate of angular momentum.

Suppose $R(r) \propto r^{-\alpha}$ and $V(r) \propto -r^{-\beta}$ near the origin. Show that $\alpha < 3/2$ is required if the wavefunction is to be normalizable, but that $\alpha < 1/2$ (or $\alpha < (3-\beta)/2$ if $\beta > 2$) is required in order for the expectation value of the energy to be finite.