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Key concepts and results

(1) Hubble's law $v = Hd$ for $v \ll c$.

(2) Hubble constant $H_0 = 100\,h\,\mathrm{km\,s^{-1}Mpc^{-1}}$.

(3) Hubble time $t_{\rm H} \equiv 9.78\,h^{-1}\mathrm{Gyr}$.

(4) Hubble length $c/H_0 = 3000\,h^{-1}\,\mathrm{Mpc}$.

(5) Scale factor and Hubble parameter $\mathbf{x}(t) = R(t)\mathbf{x}(t_0)$, so $H = \dot{R}/R$.

(6) Robertson-Walker metric

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left[dr^2 + S_k^2(r)\,d\psi^2\right], \quad \text{where}$$

$$S_k(r) = \begin{cases} \sin r & (k = 1) \\ \sinh r & (k = -1) \\ r & (k = 0). \end{cases}$$

(7) Comoving distance dr is an element of dimensionless comoving radius. Give dimensions of length via $R_0 dr$, where R_0 is current value of scale factor. $R(t)dr$ is an element of *proper* length, and is smaller than $R_0 dr$ at early times.

(8) Redshift and scale factor

$$1 + z \equiv \frac{\lambda_{\rm obs}}{\lambda_{\rm emit}} = \frac{R_0}{R(t_{\rm emit})}.$$

Often write $a(t) = R(t)/R_0$, so $a(t) = (1+z)^{-1}$.

(9) The Friedmann equation $\dot{R}^2 - 8\pi G\rho R^2/3 = -kc^2$.

(10) Critical density $\rho_c = 3H^2/8\pi G$.

(11) Density parameter $\Omega \equiv \rho/\rho_c = 8\pi G\rho/3H^2$.

(12) Current scale factor $R_0 = \frac{c}{H_0}\left[(\Omega_0 - 1)/k\right]^{-1/2}$.

(13) Conservation $\rho/\rho_0 = (R/R_0)^{-\alpha}$, where $\alpha = 4$ (radiation), 3 (matter) or 0 (vacuum), so

$$\frac{8\pi G\rho}{3} = H_0^2(\Omega_v + \Omega_m a^{-3} + \Omega_r a^{-4})$$

(using the normalized scale factor $a = R/R_0$).

(14) Zero-curvature solutions $R \propto t^{2/3}$ (matter) or $t^{1/2}$ (radiation). The $\Omega = 1$ matter-only universe is called the Einstein–de Sitter model.

(15) Effects of pressure $d[\rho c^2 R^3] = -p d[R^3]$, so

$$\ddot{R} = -4\pi G R(\rho + 3p/c^2)/3.$$

(16) Energy density of the vacuum $p_{\text{vac}} = -\rho_{\text{vac}}\, c^2$.

(17) Vacuum-dominated universe $R \propto \exp Ht$; $H = \sqrt{\frac{8\pi G\rho_v}{3}}$.

(18) Age–redshift relation Since $1 + z = R_0/R(z)$,

$$\frac{dz}{dt} = -\frac{R_0}{R^2}\frac{dR}{dt} = -(1+z)H(z),$$

Use Friedmann equation in the form $H^2 = 8\pi G\rho/3 - kc^2/R^2$. Inserting the expression for $\rho(a)$ gives

$$H^2(a) = H_0^2\left[\Omega_v + \Omega_m a^{-3} + \Omega_r a^{-4} - (\Omega - 1)a^{-2}\right],$$

hence dz/dt.

(19) Age of universe

$$H_0 t_0 \simeq \frac{2}{3}\,(0.7\Omega_m - 0.3\Omega_v + 0.3)^{-0.3}.$$

(20) Distance–redshift relation The equation of motion for a photon is $R\,dr = c\,dt$, so $R_0 dr/dz = (1+z)c\,dt/dz$, or

$$R_0\frac{dr}{dz} = \frac{c}{H(z)} = \frac{c}{H_0}\left[(1-\Omega)(1+z)^2 + \Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2} dz.$$

(21) Matter–radiation equality $1 + z_{\text{eq}} = 23\,900\,\Omega h^2\,(T/2.73\,\text{K})^{-4}$.

(22) Ultrarelativistic background Number density of quanta, $n \propto T^3$. Damped by $\sim \exp(-mc^2/kT)$ below threshold.

(23) Freezeout Interactions cease when **expansion timescale** $\gtrsim H(z)^{-1}$. Happens at $\simeq 10^{10}$ K or 1 MeV for weak interactions.

(24) Massive neutrinos $\rho = mn$, so

$$\Omega h^2 = \frac{\sum m_i}{93.5\,\mathrm{eV}}.$$

(25) Neutron freezeout $n_n/n_p = e^{-\Delta mc^2/kT} \simeq e^{-1.5(10^{10}\,\mathrm{K}/T)}$.

(26) Helium fraction

$$Y = \frac{4 \times n_n/2}{n_n + n_p} = \frac{2}{1 + n_p/n_n}$$

(neglecting neutrons in other elements). So, $Y = 0.25$ requires freezeout at $n_n/n_p \simeq 1/7$.

(27) Nucleosynthesis Deuterium formation becomes favoured at about 10^{10} K, when the universe was about 1 s old, and effectively ends in Helium when it has cooled by a factor of 10, and is about 100 times older.

(28) Nucleosynthesis baryons

$$\Omega_{\mathrm{B}} h^2 \simeq 0.02 \pm 0.002.$$

(29) Last scattering $z_{\mathrm{LS}} = 1080 \pm 80$.

(30) CMB temperature $T = 2.728 \pm 0.004\,\mathrm{K}$.

(31) Flux–luminosity relation

$$S_\nu(\nu_0) = \frac{L_\nu([1+z]\nu_0)}{4\pi R_0^2 S_k^2(r)(1+z)} = \frac{L_\nu(\nu_0)}{4\pi R_0^2 S_k^2(r)(1+z)^{1+\alpha}},$$

where the second expression assumes a power-law spectrum $L \propto \nu^{-\alpha}$.

(32) Surface brightness $I_\nu(\nu_0) = B_\nu([1+z]\nu_0)/(1+z)^3$.

(33) Distance-redshift relation

$$\begin{aligned} \textbf{angular-diameter distance:}\quad & D_{\mathrm{A}} = (1+z)^{-1} R_0 S_k(r) \\ \textbf{luminosity distance:}\quad & D_{\mathrm{L}} = (1+z)\, R_0 S_k(r). \end{aligned}$$

(34) Euclidean counts $N(>S) \propto S^{-3/2}$,

(35) Isothermal sphere $\rho \propto r^{-2} \Rightarrow V(r)$ constant.

(36) Cluster baryon fraction $M_{\mathrm{B}}/M_{\mathrm{tot}} \simeq 0.01 + 0.05h^{-3/2}$.

(37) Density perturbation $\delta(\mathbf{x}) \equiv \frac{\rho(\mathbf{x}) - \langle\rho\rangle}{\langle\rho\rangle}$.

(38) Fourier expansion $\delta(\mathbf{x}) = \sum \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}$.

(39) Power spectrum and correlation function :

$$\xi(\mathbf{r}) \equiv \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle = \sum |\delta_{\mathbf{k}}|^2 \, e^{-i\mathbf{k}\cdot\mathbf{r}}.$$

(40) Power-law spectra $\left\langle |\delta_k|^2 \right\rangle \propto k^n$. $n = 1$ is the scale-invariant spectrum (fractal metric).

(41) Perturbation growth $\delta \propto t^{2/3} \propto a(t)$ $(\Omega_m = 1)$. Potential is constant.

(42) Types of non-baryonic dark matter HDM decouples late while relativistic; WDM decouples early while relativistic; CDM decouples early while non-relativistic; CDM more massive as number density is reduced by annihilation.

(43) Transfer function Break in spectrum at $D_{\rm H}(z_{\rm eq}) \simeq 16\,(\Omega_m h^2)^{-1}$Mpc.

(44) The horizon problem $r_{\rm H} = \int_0^t \frac{c\,dt}{R(t)}$. The standard radiation-dominated $R \propto t^{1/2}$ law makes this integral converge near $t = 0$. Causal contact needs $R \propto t^\alpha$, with $\alpha > 1$: an accelerating universe.

(45) Amount of inflation needed $\Delta t_{\rm inflation} > 60\, H^{-1}_{\rm inflation}$.

(46) Inflationary fluctuations $\delta t = \frac{\delta\phi}{\dot\phi}$.

(47) Anisotropies in the CMB

$$\frac{\delta T}{T} \sim \frac{\delta\Phi}{c^2} \quad \text{(gravity)}; \qquad \frac{\delta T}{T} \sim \frac{1}{3}\frac{\delta\rho}{\rho} \quad \text{(adiabatic)}.$$

(48) Horizon at last scattering $D_{\rm H}(z) \equiv R_0 \int_z^\infty dr = \frac{2c}{H_0}\,[(1+z)\,\Omega_m]^{-1/2} = 181\,\Omega_m^{-1/2}\,h^{-1}\,$Mpc

(49) Current horizon size

$$D_{\rm H} = \frac{2c}{\Omega_m H_0} \quad \text{(open)}$$
$$D_{\rm H} \simeq \frac{2c}{\Omega_m^{0.4} H_0} \quad \text{(flat)};$$

(50) Angular size of horizon at last scattering $\theta = 1.8\,\Omega_m^{1/2}$ degrees.