Cosmological dynamics

Gravity and spherical bodies

If the size of the universe is proportional to the scale factor, $R(t)$, how do we determine how this varies with epoch? This sounds a nasty problem given the complexities of curved space, but in fact the tools needed are very simple ones.

Newton was the first to understand two special properties of gravity. One is familiar from everyday experience: the Earth's gravity appears to pull towards its centre of mass, as if all the mass was concentrated there. Look at the **inverse-square law** for the gravitational acceleration, a, towards a body of mass M at a distance r:

$$\boxed{a = GM \,/\, r^2,}$$

where G is the gravitational constant. This expression says nothing about the size of the body, and Newton was able to probe that this equation gave the right answer for a spherical body of any radius up to r.

If the body becomes larger than r, we have a combination of a smaller body and a spherical shell. Newton's second achievement was to show that the force inside any uniform spherical shell vanishes: to a particle inside the shell, the mass might as well not be there. So, if you could excavate a little hole at the centre of the Earth, you would float weightless there, just as in deep space.

Friedmann's equation

These properties of gravity are all the tools that are needed to derive the dynamics of the expanding universe. This was first done in 1922 by the Russian mathematician **Friedmann**; sadly, Einstein thought he was wrong, and Friedmann died in 1925, before the observational fact of the Hubble expansion was established. Friedmann used a highly algebraic argument in relativity, and would probably have been surprised that the simple argument used here works.

In any case, imagine a small spherical region of the universe, of radius $R(t)$. This sounds like it has to be the whole universe, but the scale factor is just *proportional* to the size of the universe, so we can always choose it to be small at a given time. A galaxy on the edge of this ball is pulled towards the centre by gravity, but is unaffected by the galaxies at larger radii. The mass within the ball is the product of density and volume: $M = \rho \times 4\pi R^3/3$. This mass stays constant as the ball expands. Notice that this requires the following dependence of density on the size of the universe:

$$\boxed{\rho \propto R(t)^{-3}.}$$

Not surprisingly, the matter becomes more tenuous as the universe expands.

The problem is now amazingly simple: the equation of motion for the marker galaxy at radius R is just the same as that for a cannonball moving in the gravitational field of the Earth. We solve this via conservation of energy: kinetic energy ($mv^2/2$) plus gravitational potential energy ($-GM/R$) is a constant. The mass of the galaxy, m, appears in both terms, so it can be factored out and absorbed into the definition of the constant. This gives the equation

$$\frac{v^2}{2} - \frac{GM}{R} = \text{constant} = \frac{v^2}{2} - \frac{4\pi G\rho R^3}{R}.$$

Finally, we can use Hubble's law to write $v = HR$, and multiply by $2/R^2$ (absorbing the factor of 2 by redefining the constant) to get **Friedmann's equation**:

$$\boxed{H^2 - 8\pi G\rho/3 = \text{constant}/R(t)^2.}$$

It is astonishing to think that the whole universe, perhaps infinite in extent, is compelled to obey this simple equation. To get it, we have only cheated slightly: Newton's results were based on the inverse-square law, and we might worry about using this in a curved space. Fortunately, **Birkhoff's theorem** says that Newton's results carry over exactly into general relativity.

From the point of view of the scale factor, what we have called the big bang now corresponds to asking whether the solution of Friedmann's equation has a **singularity** – i.e. whether $R(t)$ goes to zero at $t = 0$. We will see later that this singular behaviour does exist: a region of space that is now billions of light-years across originated from a tiny volume that you could hold in your hand. However, this doesn't have to mean that the whole universe becomes concentrated into a single point: if the universe is open, there will still be an infinity of space and matter at larger distances. It's best to think of the singularity as a point of infinite density, rather than zero size.

The density parameter

Having got Friedmann's equation, what does it mean? There is an unknown constant, which represents the total energy. There is a **critical density** at which this vanishes:

$$\boxed{\rho_{\rm crit} = 3H^2/(8\pi G) = 0.79 \times 10^{-26}\,\mathrm{kg\,m^{-3}},}$$

or roughly just one atom per cubic metre! This is the density below which the universe has **escape velocity**; a particle thrown from the surface of the Earth at about 4 miles per second will escape from the Earth's gravity entirely, and the analogue of this in cosmology is that $R(t)$ increases without limit. On the other hand, if the density exceeds the critical value, the universe behaves like a ball thrown upwards with only moderate speed: gravity will eventually halt the expansion, and the universe will then start to contract, heading towards a

big crunch as the density rises without limit. Because of the important role played by the critical density, it is very common to define a dimensionless **density parameter**:

$$\boxed{\Omega = \rho \, / \, \rho_{\rm crit}.}$$

Ω and h are the two numbers that all cosmologists want to be able to measure.

This Newtonian discussion has missed one very important thing, which can only be proved using general relativity: the constant in Friedmann's equation is related to the global geometry of the universe:

$$\boxed{\begin{array}{ll} \Omega > 1 : & \text{closed universe} \\ \Omega < 1 : & \text{open universe} \end{array}}$$

It is reasonable enough that more mass makes space more curved, but it is a very deep coincidence that the critical density for expansion coincides with that for curvature. A closed universe is also the one that will re-collapse; not a nice thought if you suffer from claustrophobia.

Observational cosmology

How are we to measure Ω? We will describe the main methods later in some detail. One is direct: try to measure the local mass density. The other depends on the way in which the expansion of the universe affects the appearance of distant objects. At small distances, we can ignore the curvature of space and just use the **inverse-square law for flux**:

$$\boxed{f = \frac{L}{4\pi r^2}; \quad r = \frac{cz}{H},}$$

where f is the flux density and L is the luminosity. For large redshifts, this simple linear relation is not applicable. This is partly due to the curvature of space, but partly due to the acceleration of the expansion. If gravity is slowing the expansion down very strongly, then we see a more rapidly expanding universe when we look to large distances (i.e. back in time). At a given distance, the redshift will therefore be larger for a greater deceleration (greater Ω); alternatively, for a given redshift, the distance will be smaller and so the object will appear *brighter*. This fact will be used later to weigh the universe.

The radiation era

So far, we have only considered a universe containing simple forms of matter, whose density scales as $\rho \propto R^{-3}$ (e.g. rocks). One important exception to this is radiation. Suppose we fill the universe with photons of a given energy, E. Because of the redshift, this energy falls as $E \propto 1/R$; through $E = mc^2$, this means that the mass of each photon falls as well. The radiation density is the energy times the photon number density, which scales as R^{-3}. Overall, we get the faster behaviour

$$\boxed{\rho_{\rm radiation} \propto R^{-4}.}$$

Suppose the universe contains a tiny amount of radiation now: perhaps $\rho_{\rm radiation}/\rho_{\rm matter} = 10^{-10}$. Nevertheless, the different power-law behaviours of the density mean that the radiation was more important in the past. The ratio was 10^{-9} when the universe was 10 times smaller ($z = 9$), etc. Eventually, the two densities must cross over, so there is always a **radiation-dominated era** at early times.

Friedmann's equation turns out still to apply, independent of the nature of the mass – just add the radiation to ρ and allow for its different dependence on scale factor. We won't go into how to solve the equation, since this needs calculus, but it is worth quoting the answer for the time dependence of the the scale factor, and how this depends on the type of matter:

$$\boxed{\begin{array}{ll} R \propto t^{2/3} & \text{(matter dominated)} \\ R \propto t^{1/2} & \text{(radiation dominated).} \end{array}}$$

These results assume $\Omega = 1$.

The horizon

An important question is: how much of the expanding universe can we actually observe? If the universe is now 13 billion years old, then light can only have travelled a distance $d = ct$, i.e. 13 billion light years or about 4000 Mpc. This distance corresponds to an infinite observed redshift, and objects at greater distances will be invisible.

You might think this argument is wrong. After all, the universe was smaller in the past; doesn't this allow light to get from object to object more easily? After all, what is now 4000 Mpc was once only 1 cm if you go to early enough times. In fact, this doesn't change things, because the universe expands very fast at early times; a photon that starts off at $t = 0$ doesn't catch up with much material, because it is receding at nearly the speed of light.

This means that the cosmological horizon was very small at early times: the presently-observable universe was divided into zones that could not communicate with each other until now. This turns out to be an important clue to how the expansion got started.