Astrophysical Cosmology

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Lecture 1
The large-scale distribution of galaxies
Temperature Variations in the Cosmic Microwave Background
Properties of the Universe

- Universe is expanding.
- Components of the Universe are:
  - Dark Energy: 73%
  - Cold Dark Matter: 23%
  - Atoms: 4%
- Universe is 13.7 Billion years old.
- Expansion is currently accelerating.
1920’s: The Great Debate

Are these nearby clouds of gas?

Or distant stellar systems (galaxies)
In 1924 Edwin Hubble finds Cepheid Variable stars in M31. Cepheid intrinsic brightness correlate with variability (standard candle), so can measure their distance. Measured 3 million light years (1Mpc) to M31.
The Expanding Universe

- Between 1912 and 1920 Vesto Slipher finds most galaxy’s spectra are redshifted.

Slipher is first to suggest the Universe is expanding!
In 1929 Hubble also finds fainter galaxies are more redshifted. Infers that recession velocities increase with distance.

Hubble’s Law

\[ V \propto D \]

\[ V = HD \]
The Expanding Universe

Distance = Velocity x Time

\[ V = HD, \quad H = \frac{1}{t} \]

1. Grenade Model:
The Expanding Universe

Distance = Velocity x Time

\[ V = H \cdot D, \quad H = \frac{1}{t} \]

2. Scaling Model:

\[ x(t) = R(t) \cdot x_0 \]
The Expanding Universe

Distance = Velocity x Time

\[ V = HD, \quad H = 1/t \]

Hubble Time:
\[ t_H = 1/H \]

\[ H = 70 \text{ km/s/Mpc} \]
\[ t_H = 14 \text{ Gyrs} \]
In 1915 Albert Einstein showed that the geometry of spacetime is shaped by the mass-energy distribution. General Theory of Relativity required to describe the evolution of spacetime.
Cosmological Coordinates \((t, x)\):
- How do we lay down a global coordinate system?
- In general we cannot.

Can we lay down a local coordinate system?
- Yes, can use Special Relativity locally, if we can cancel gravity.

We can cancel gravity by free-falling (equivalence principle).
Relativistic Cosmologies

• Equivalence Principle:
Relativistic Cosmologies

- Equivalence Principle:
Relativistic Cosmologies

• In free-fall, a **Fundamental Observer** locally measures the spacetime of Special Relativity.

• Special Relativity Minkowski-space line element:

\[-ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2\]

• So all Fundamental Observers will measure time changing at the same rate, \(dt\).

• **Universal cosmological time coordinate**, \(t\).
Relativistic Cosmologies

• How can we synchronize this Universal cosmological time coordinate, t, everywhere?

• With a Symmetry Principle:
  • On large-scales Universe seems isotropic (same in all directions, eg, Hubble expansion, galaxy distribution, CMB).
  • Combine with Copernican Principle (we’re not in a special place).
Relativistic Cosmologies

- Isotropy + Copernican Principle = homogeneity (same in all places)

\[ \rho = \rho_0 \]

So uniform density everywhere
Relativistic Cosmologies

- Isotropy + homogeneity = Cosmological Principle

\[ \rho_2 = \rho_1 = \rho_0. \]

So uniform density everywhere
Relativistic Cosmologies

• With the Cosmological Principle, we have uniform density everywhere.

• Density will decrease with expansion, so $\rho = \rho(t)$.

• So can synchronize all Fundamental Observers clocks at pre-set density, $\rho_0$, and time, $t_0$:

  $$t_0 : \rho(t_0) = \rho_0$$
Relativistic Cosmologies

• What is the line element (metric) of a relativistic cosmology?

• Locally Minkowski line element (Special Relativity):

\[- ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \]

• Spacetime Diagram:
Relativistic Cosmologies

• A general line element (Pythagoras on curved surface):

\[ c^2 d\tau^2 = g_{\nu\mu} dx^\nu dx^\mu \quad x^\mu = (ct, x, y, z) \]

• Minkowski metric tensor:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

• We have Universal Cosmic Time of Special Relativity, t, so

\[ c^2 d\tau^2 = c^2 dt^2 - d\sigma_3^2 \]

\[ \sigma_3^2 \quad \text{Spatial part of metric} \]
• What is spatial metric, $\sigma_3^2$?

• From Cosmological Principle (homogeneity + isotropy) spatial curvature must be constant everywhere.

• Only 3 possibilities:
  • Sphere – positive curvature
  • Saddle – negative curvature
  • Flat – zero curvature.
• What is form of $\sigma_3^2$?

• Consider the metric on a 2-sphere of radius $R$, $\sigma_2^2$:

$$d\sigma_2^2 = R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)$$
Relativistic Cosmologies

- The metric on a 2-sphere of radius R:

\[ d\sigma_2^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

- Now re-label \( \theta \) as \( r \) and \( \phi \) as \( \theta \):

\[ d\sigma_2^2 = R^2 (dr^2 + \sin^2 r d\theta^2) \]

where \( r = (0, \pi) \) is a dimensionless distance.
• Can generate other 2 models from the 2-sphere:

\[ d\sigma_2^2 = R^2 (dr^2 + \sin^2 r \, d\theta^2) \]

\[ r \to ir, \, R \to iR \]

\[ d\sigma_2^2 = R^2 (dr^2 + \sinh^2 r \, d\theta^2) \]

\[ r \ll R \]

\[ d\sigma_2^2 = R^2 (dr^2 + r^2 \, d\theta^2) \]
Relativistic Cosmologies

• General 3-metric for 3 curvatures:

\[
d\sigma^2 = R^2 (dr^2 + S_k^2(r)d\theta^2) = \begin{cases} 
\sin(r), & k = +1 \\
r, & k = 0 \\
\sinh(r), & k = -1 
\end{cases}
\]
• Different properties of triangles on curved surfaces:
Different properties of triangles on curved surfaces:

\[ d \sigma_2^2 = R^2 (dr^2 + S_k^2 (r)d\theta^2) \]

\[ S_k (r) = \begin{cases} 
\sin (r), & k = +1 \\
 r, & k = 0 \\
\sinh (r), & k = -1 
\end{cases} \]
Finally add extra compact dimension:

\[ d\theta^2 \rightarrow d\theta^2 + \sin^2 \theta d\phi^2 \]

Promote a 2-sphere to a 3-sphere

\[ dr^2 + S_k^2(r)d\theta^2 \rightarrow dr^2 + S_k^2(r)(d\theta^2 + \sin^2 \theta d\phi^2) \]

So metric of 3-sphere is

\[ d\sigma^2_3 = R^2(dr^2 + S_k^2(r)d\psi^2), \]
\[ d\psi^2 = d\sigma^2_2 = d\theta^2 + \sin^2 \theta d\phi^2 \]
• The Robertson-Walker metric generalizes the Minkowski line element for symmetric cosmologies:

\[ c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\psi^2 \]

- The Robertson-Walker Metric
Relativistic Cosmologies

• Alternative form of the Robertson-Walker metric:

\[
c^2 d\tau^2 = c^2 dt^2 - R^2 (dr^2 + \sin^2(r) d\psi^2)
\]

\[
= c^2 dt^2 - R^2 \left( \frac{dy^2}{1 - y^2} + y d\psi^2 \right)
\]

\[
y = \sin r
\]

\[
\frac{d \sin^{-1} y}{dy} = \frac{1}{\sqrt{1 - y^2}}
\]
Relativistic Cosmologies

• Alternative form of the Robertson-Walker metric:

\[ c^2 \, d \tau^2 = c^2 \, dt^2 - R^2 (dr^2 + S_k^2(r) \, d\psi^2) \]

\[ = c^2 \, dt^2 - R^2 \left( \frac{dy^2}{1 - ky^2} + y \, d\psi^2 \right) \]

\[ y = S_k(r) \]

\[ \frac{dS_k^{-1}y}{dy} = \frac{1}{\sqrt{1 - ky^2}} \]
Relativistic Cosmologies

• The Robertson-Walker models.

\[ c^2 d\tau^2 = c^2 dt^2 - R^2 (dr^2 + S_k^2(r) d\psi^2) \]

- \( k = +1 \): positive curvature everywhere, spatially closed, finite volume, unbounded.
- \( k = -1 \): negative curvature everywhere, spatially open, infinite volume, unbounded.
- \( k = 0 \): flat space, spatially open, infinite volume, unbounded.
Relativistic Cosmologies

• The Robertson-Walker models.

• We have defined the comoving radial distance, \( r \), to be dimensionless.

• The current comoving angular distance is:
  \[ d = R_0 S_k(r) \text{ (Mpc)}. \]

• The proper physical angular distance is:
  \[ d(t) = R(t) S_k(r) \text{ (Mpc)}. \]
Lecture 3
Relativistic Cosmologies

• Superluminal expansion:
  The proper radial distance is

$$d(t) = R(t)r$$

The proper recession velocity is:

$$v(t) = \dot{d}(t) = \dot{R}(t)r = \frac{\dot{R}}{R}d > c$$

What does this mean?
Locally things are not moving (just Special Relativity).
But distance (geometry) is changing.
No superluminal information exchange.
Light Propagation

• How does light propagate through the expanding Universe?

• Let a photon travel from the pole (r=0) along a line of constant longitude (dθ=0, dφ=0).

• The line element for a photon is a null geodesic (zero proper time):

\[ c^2 d\tau^2 = c^2 dt^2 - R^2(t)dr^2 = 0 \]
Light Propagation

• Equation of motion of a photon:

\[ c^2 dt^2 = R^2(t) dr^2 \]

\[ r(t) = \int_0^t \frac{c dt}{R(t)} \]

The comoving distance light travels.
Light Propagation

• Let’s assume \( R(t) = R_0(t/t_0)^\alpha \): 

\[
r(t) = \int_0^t \frac{cdt'}{R(t')}
\]

\[
= \frac{c t_0^\alpha}{R_0} \int_0^t (t')^{-\alpha} dt'
\]

\[
= \frac{c t_0^\alpha}{R_0} \left[ \frac{t^{1-\alpha}}{1-\alpha} \right]_0^t, \quad \alpha \neq 1
\]
Causal structure

• Lets assume $\alpha > 1$:

$$R(t) = R_0 \left( \frac{t}{t_0} \right)^2 :$$

$$r(t) = \frac{ct^2}{R_0} \left( \frac{1}{t_1} - \frac{1}{t} \right)$$

$$l(t) = R(t)r(t) = c \left( \frac{t^2}{t_1} - t \right)$$

For $t \gg t_1$, $r$ is constant. This is called an **Event Horizon**. As $t_1$ tends to 0, $l(t)$ diverges, everywhere is causally connected.
\[ R(t) = R_0 \left( \frac{t}{t_0} \right)^{1/2} : \]

\[ r(t) = 2c \left( \frac{t_0^{1/2}}{R_0} \right) t^{1/2} \]

\[ l(t) = R(t) r(t) = 2ct \]

At early times all points are causally disconnected. The furthest that light can have travelled is called the \textbf{Particle Horizon}.
Cosmological Redshifts

• Consider the emission and observation of light:

\[
\begin{align*}
& t = t_0 \\
& r = 0 \\
& d = Rr = 0 \\
& t = t_1 \\
& r = r_1 = \int_{t_0}^{t_1} \frac{dt}{R(t)} \\
& d_1 = R(t_1)r_1
\end{align*}
\]
Cosmological Redshifts

- Consider the emission and observation of light:

\[
\begin{align*}
\text{t} &= t_0 \\
\text{r} &= 0 \\
\text{d} &= R \text{r} = 0
\end{align*}
\]

A bit later:

\[
\begin{align*}
\text{t} &= t_0 + \delta t_0 \\
\text{r} &= 0 \\
\text{d} &= R \text{r} = 0
\end{align*}
\]

\[
\begin{align*}
\text{t} &= t_1 \\
\text{r} &= r_1 = \int_{t_0}^{t_1} \frac{dt}{R(t)} \\
\text{d} &= R(t_1) r_1
\end{align*}
\]

\[
\begin{align*}
\text{t} &= t_1 + \delta t_1 \\
\text{r} &= r_1 = \int_{t_0 + \delta t_0}^{t_1 + \delta t_1} \frac{dt}{R(t)} \\
\text{d} &= R(t_1 + \delta t_1) r_1
\end{align*}
\]
Cosmological Redshifts

- But the comoving position of an observer is a constant:

\[ r' = \int_{t_0}^{t_1} \frac{dt}{R(t)} = \int_{t_0 + \delta t_0}^{t_1 + \delta t_1} \frac{dt}{R(t)} = \int_{t_0}^{t_1} \frac{dt}{R(t)} + \left( \int_{t_1}^{t_1 + \delta t_1} \frac{dt}{R(t)} - \int_{t_0}^{t_0 + \delta t_0} \frac{dt}{R(t)} \right) \]

\[ \frac{\delta t_1}{R(t_1)} = \frac{\delta t_0}{R(t_0)} \]

Say the wavelength of light is \( \lambda = c\delta t \):

\[ \frac{\lambda_0}{R_0} = \frac{\lambda_1}{R_1} \]

so

\[ (1 + z) \equiv \frac{v_0}{v_1} = \frac{R_1}{R_0} \]
Cosmological Redshifts

• Can also understand as a series of small Doppler shifts:

\[ \delta v = \frac{c}{\nu} \delta V = -H \delta t = -\frac{\dot{R}}{R} \delta t = -\frac{\delta R}{R} \]

\[ \nu \propto R^{-1} \]
Decay of particle momentum

- Every particle has a de Broglie wavelength:
- So momentum (seen by FO’s) is redshifted too:

\[ p = \frac{h}{\lambda} \Rightarrow p \propto \frac{1}{R} \]

- Why? (“Hubble drag”, “expansion of space”?)

\[ d = Rr, \ V = Hd \]

\[ t = 0 \quad \text{and} \quad t = \delta t \]
Lecture 4
The Dynamics of the Expansion

In 1922 Russian physicist Alexandre Friedmann predicted the expansion of the Universe.

Newtonian Derivation:

Total Energy = K.E. + P.E

\[ E_{TOT} = \frac{1}{2} m (\dot{R}r)^2 - \frac{GMm}{Rr} \]

Birkhoff’s Theorem

\[ V = \dot{R}r \]

\[ M = 4\pi\rho(Rr)^3/3 \]
The Dynamics of the Expansion

In 1922 Russian physicist Alexandre Friedmann predicted the expansion of the Universe.

Friedmann Equation:

\[ H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{4\pi G \rho}{3} - \frac{k c^2}{R^2} \]

Birkhoff’s Theorem

\[ M = 4\pi \rho (Rr)^3 / 3 \]
Geometry & Density

• There is a direct connection between density & geometry:

\[
\dot{R}^2 = \frac{8\pi G \rho}{3} R^2 - kc^2
\]

- \(\rho R^2 \to 0\)
- \(\dot{R}^2 = -kc^2\)
- \(k = -1\)
- \(R = ct\)

• So a low-density model will evolve to an empty, flat expanding universe.
Geometry & Density

• There is a direct connection between density & geometry:

\[ H^2 = \frac{8\pi G \rho}{3} - \frac{c^2}{R^2} \]

\[ R \rightarrow \infty \]

\[ H^2 = \frac{8\pi G \rho}{3} \]

\[ k = 0 \]

• So with the right balance between \( H \) and \( \rho \), we have a flat model.
We can define a critical density for flat models and hence a density parameter which fixes the geometry.

- For $k = +1$, $\rho > \rho_c$, $\Omega > 1$
- For $k = -1$, $\rho < \rho_c$, $\Omega < 1$
- For $k = 0$, $\rho = \rho_c$, $\Omega = 1$

Critical density: $\rho_c = \frac{3H^2}{8\pi G}$

Density parameter: $\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G \rho}{3H^2}$
Critical density & density parameter

- How does $\Omega$ evolve with time?

\[
H^2 = \frac{8\pi G \rho}{3} - \frac{k c^2}{R^2} \\
1 = \frac{8\pi G \rho}{3 H^2} - \frac{k c^2}{R^2 H^2} \\
\Omega(t) = 1 + \frac{k c^2}{R^2(t) H^2(t)}
\]
What is present curvature length?

Define a dimensionless Hubble parameter:

\[ h = \frac{H}{100 \text{ kms}^{-1} \text{Mpc}^{-1}} \]

\[ R_0 = \frac{c}{H_0} \left[ \frac{(\Omega - 1)}{k} \right]^{-1/2} = 3000 \left[ \frac{(\Omega - 1)}{k} \right]^{-1/2} \text{ h}^{-1} \text{Mpc} \]
What is present density?

\[ \rho_0 = 1.88 \times 10^{-26} \left( \Omega h^2 \right) \text{ kg m}^{-3} \]

\[ = 2.78 \times 10^{11} \left( \Omega h^2 \right) \text{M}_{\text{SUN}} \text{ Mpc}^{-1} \]

Or 1 small galaxy per cubic Mpc.
Or 1 proton per cubic meter.
The meaning of the expansion of space

• Consider an expanding empty, spatially flat universe. c.f. a relativistic Grenade Model:

• Minkowski metric:

\[ c^2 d\tau^2 = c^2 dt^2 - (dr^2 + r^2 d\psi^2) \]

• Let \( v = Hr, H = 1/t \) so \( v = r/t \).

• Switch to comoving frame:

\[ t' = t / \gamma = t \sqrt{1 - \left(\frac{v}{c}\right)^2} = t \sqrt{1 - \left(\frac{r}{tc}\right)^2} \]
The meaning of the expansion of space

• Rewrite in terms of $t'$ (comoving time):

$$\gamma^2 = 1 + \left( \frac{r}{ct'} \right)^2$$

• Hence in the comoving frame:

$$c^2 \, d\tau^2 = c^2 \, dt'^2 - \left( \frac{dr^2}{1 + (r/ct')^2} + r^2 \, d\psi^2 \right)$$

• but this is a $k=-1$ open model with $R=ct$!

• So what is curvature?

• And is space expanding?
The matter dominated universe

- Consider a universe with pressureless matter (dust, galaxies, or cold dark matter).
- As Universe expands, density of matter decreases: \( \rho = \rho_0 (R/R_0)^{-3} \).
- Consider a flat model: \( k=0, \Omega=1 \).

\[ H^2 = \frac{8\pi G \rho}{3} \]

\[ R \propto t^{2/3} \]
The matter dominated universe

- The spatially flat, matter-dominated model is called the Einstein-de Sitter model.

\[ R \propto t^{2/3} \]

\[ H_0 = \frac{\dot{R}}{R} = \frac{2}{3t_0} \]

\[ t_0 = \frac{2}{3H_0} = 9.3 \text{ Gyrs} \]
The matter dominated universe

- Consider an open or closed, matter-dominated universe.
- Define a conformal time, $d\eta = c dt / R(t)$.

$$c^2 d\tau^2 = R^2(t) [d\eta^2 - (dr^2 + S_k^2(r) d\psi^2)]$$
The matter dominated universe

- Consider a closed, matter-dominated universe.
- Define a conformal time, $d\eta = cdt/R(t)$.

$$\left( \frac{R'}{R_*} \right)^2 = 2 \left( \frac{R}{R_*} \right) - \left( \frac{R}{R_*} \right)^2$$

$$R(\eta) = R_* (1 - \cos \eta)$$

$$ct(\eta) = R_* (\eta - \sin \eta)$$

$$R_* = \frac{4 \pi G \rho_0 R_0^3}{3 c^2}$$
The matter dominated universe

- Consider an open or closed, matter-dominated universe.
- Define a conformal time, \( d\eta = c dt / R(t) \).

\[
\left( \frac{R'}{R_*} \right)^2 = 2 \left( \frac{R}{R_*} \right) - k \left( \frac{R}{R_*} \right)^2
\]

\[
R(\eta) = kR_*(1 - C_k \eta)
\]

\[
ct(\eta) = kR_*(\eta - S_k \eta)
\]

\[
R_* = \frac{4 \pi G \rho_0 R_0^3}{3 c^2}
\]
The matter dominated universe

- So for matter-dominated models geometry/density=fate.

```
\Omega < 1
k = -1
Expand forever

\Omega > 1
k = +1
Eventual recollapse

\Omega = 1
k = 0
```

\text{R(t)}

\text{``Big Bang''}

\text{``Big Crunch''}

\text{time t}
The radiation dominated universe

- As Universe expands, density of matter decreases:
  \[ \rho_m = \rho_{0m} \left( \frac{R}{R_0} \right)^{-3}. \]
- Radiation energy density:
  \[ \rho_r = \rho_{0r} \left( \frac{R}{R_0} \right)^{-4}. \]
- At early enough times we have radiation-dominated Universe.

For \( T(\text{CMB}) = 2.73\,\text{K} \), \( z_{eq} = 1000 \).
The radiation dominated universe

• At early enough times we also have a flat model: $k=0$

$$\dot{R}^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R_0}{R} \right)^3 + \rho_{\gamma,0} \left( \frac{R_0}{R} \right)^4 \right) \dot{R}^2 + c^2 k$$

$$R \to 0, \quad \Omega \to 1$$

$$\dot{R}^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R_0}{R} \right)^3 + \rho_{\gamma,0} \left( \frac{R_0}{R} \right)^4 \right) R^2$$

$$\dot{R}^2 = \frac{8\pi G}{3} \left( \rho_{\gamma,0} R_0^4 \right) R^{-2}, \quad \Rightarrow \quad R \propto t^{1/2}$$

So Particle Horizon!
The radiation dominated universe

• Timescales:

• Matter-dominated: \( R \sim t^{2/3} \)

\[
H = \frac{2}{3t} = \sqrt{\frac{8\pi G \rho_m}{3}} \Rightarrow t = \frac{1}{\sqrt[3]{6\pi G \rho_m}}
\]

• Radiation dominated: \( R \sim t^{1/2} \)

\[
H = \frac{1}{2t} = \sqrt{\frac{8\pi G \rho_\gamma}{3}} \Rightarrow t = \frac{1}{\sqrt[3]{32\pi G \rho_\gamma}}
\]
The radiation dominated universe

• Spatial flatness at early times:

Recall:

\[ \Omega(t) = 1 + \frac{kc^2}{(RH)^2} = 1 + k \left( \frac{t}{t_0} \right) \]

How close to 1 can this be? At Planck time \( t = 10^{-43} \text{s} \)?

\[ \Omega(t) = 1 \pm \frac{t_{pl}}{t_0} = 1 \pm 10^{-60} \]
Energy density and Pressure

• Thermodynamics and Special Relativity:

\[ dE = -pdV \]
\[ d(\rho R^3 c^2) = -pd(R^3) \]
\[ \dot{\rho} + 3H(\rho + p / c^2) = 0 \]

• So energy-density changes due to expansion.

\[ \dot{\rho} + 3H\rho = -3Hp / c^2 \]
Energy density and Pressure

• Conservation of energy: \( \dot{\rho} = -3H(\rho + p / c^2) \)

• For pressureless matter (CDM, dust, galaxies):
  
  \[ p = 0 \quad \Rightarrow \quad \rho \propto R^{-3} \]

• Radiation pressure:
  
  \[ \rho_\gamma \propto R^4 \quad \Rightarrow \quad \dot{\rho}_\gamma = 4H\rho_\gamma \]
  
  \[ p_\gamma = \frac{1}{3}\rho_\gamma c^2 \]

• Cf. electromagnetism.
Lecture 6
Pressure and Acceleration

• Time derivative of Friedmann equation:

\[
\frac{d}{dt} \dot{R}^2 = \frac{d}{dt} \left( \frac{8\pi G \rho}{3} R^2 - kc^2 \right)
\]

\[
2 \ddot{R} \dot{R} = \frac{8\pi G}{3} (\dot{\rho} R^2 + 2 \rho \dot{R} \ddot{R})
\]

\[
\dot{\rho} = -3H(\rho + p / c^2)
\]

• Acceleration equation for R:

\[
\ddot{R} = -\frac{4\pi G}{3} \left( \rho + 3 \frac{p}{c^2} \right) R
\]
Vacuum energy and acceleration

- Gravity responds to all energy.
- What about energy of the vacuum?

- Two possibilities:
  1. Einstein’s cosmological constant.
  2. Zero-point energy of virtual particles.
Einstein’s Cosmological Constant

\[ H^2 = \frac{8\pi G \rho}{3} - \frac{k c^2}{R^2} + \frac{\Lambda}{3} \]

Einstein introduced constant to make Universe static.
Einstein’s Cosmological Constant

• Problem goes back to Newton (1670’s).

\[ \nabla^2 \Phi = 4\pi G \rho \quad \Rightarrow \quad \Phi = 4\pi G \rho r^2 \quad \Rightarrow \quad g = \nabla \Phi \propto r \]

• Einstein’s 1917 solution:

\[ \nabla^2 \Phi + \lambda \Phi = 4\pi G \rho \quad \Rightarrow \quad \Phi = \frac{4\pi G \rho}{\lambda} \quad \Rightarrow \quad g = 0 \]
Einstein’s Cosmological Constant

• But this is not stable to expansion/contraction.

Einstein called this: “My greatest blunder.”
British physicist Paul Dirac predicted antiparticles.

Werner Heisenberg’s Uncertainty Principle: Vacuum is filled with virtual particles.

Observable (Casmir Effect) for electromagnetism.
The Vacuum Energy Problem

• So Quantum Physics predicts vacuum energy.
• But summation diverges.
• If we cut summation at Planck energy it predicts an energy \(10^{120}\) times too big.
  
  Density of Universe = 10 atoms/m\(^3\)
  Density predicted = 1 million x mass of the Universe/m\(^3\)

• Perhaps the most inaccurate prediction in science? Or is it right?
Vacuum energy

- Vacuum energy is a constant everywhere: $\rho_V \sim R^0$
- Thermodynamics: Consider a piston:

\[
d\left(\rho_V c^2 R^3\right) = \rho_V c^2 d\left(R^3\right) = -p_V d\left(R^3\right)\]

\[
p_V = -\rho_V c^2
\]

The equation of state of the vacuum.
Vacuum energy and acceleration

• Effect of negative pressure on acceleration:

\[ \ddot{R} \propto -(\rho_V + 3\frac{p_V}{c^2}) = +2\rho_V \]

• So vacuum energy leads to acceleration.

\[ H^2 \propto \rho_V = \text{const} \]
\[ \dot{R} = HR \]
\[ R = R_0 e^{Ht} \]

• Eddington: \( \Lambda \) is the cause of the expansion.
General equation of State

• In general should include all contributions to energy-density.

\[ H^2 = \frac{8\pi G}{3} \rho + \frac{c^2 k}{R^2}, \quad a = \frac{R(t)}{R_0} \]

\[ = H_0^2 \left( \Omega_V + \Omega_{m,0} a^{-3} + \Omega_{\gamma,0} a^{-4} + (1 - \Omega_V - \Omega_{m,0} - \Omega_{\gamma,0}) a^{-2} \right) \]
General equation of State

- In general must solve F.E. numerically.
- Geometry is still governed by total density:
General equation of State

• In general must solve F.E. numerically.
• But in general no geometry-fate relation:
General equation of State

No singularity

EXPAND FOREVER

CLOSED

OPEN

RECOLLAPSE
Age and size of Universe

• Evolution of redshift:

\[ (1 + z) = \frac{R_0}{R(t)} \]

\[ \Rightarrow \quad \frac{dz}{dt} = -R_0 \frac{\dot{R}}{R^2} = -(1 + z)H(z) \]

where

\[ H^2(z) = H_0^2 \left( \Omega_V + \Omega_{m,o} (1+z)^3 + \Omega_{\gamma,o} (1+z)^4 + (1 - \Omega_V - \Omega_{m,o} - \Omega_{\gamma,o})(1+z)^2 \right) \]

Age of the universe:

\[ t(z) = \int_0^\infty \frac{dz}{(1 + z)H(z)} \]
Age and size of Universe

• Usually evaluate $t_0$ numerically, but approximately:

$$t_0 \approx \frac{2}{3H_0} \left[0.7\Omega_m - 0.3(\Omega_v - 1)\right]^{-0.3}$$
Age and size of Universe

- Comoving distance-redshift relation: $dr = cdt/R = cdz/R_0 H(z)$.

\[
\begin{align*}
\Omega_m &= 1 \\
R_0 r &= \frac{2c}{H_0} \left(1 - (1 + z)^{-1/2}\right) \\
\Omega_m &= 0 \\
R_0 r &= \frac{c}{2H_0} \left((1 + z) - (1 + z)^{-1}\right)
\end{align*}
\]

\[w = -1, \quad \Omega_v = 1, \quad R_0 r = cz/H_0\]
Age and size of Universe

- Comoving distance-redshift relation: \( dr = cdt/R = cdz/R_0H(z) \).

Einstein de Sitter: \( \Omega_m = 1 \)

de Sitter: \( \Omega_V = 1 \)

\[
R_0r = \frac{2c}{H_0} \left( 1 - (1 + z)^{-1/2} \right)
\]

\[
R_0r = \frac{c}{H_0} z
\]
Observational Cosmology

• Size and Volume:
  • Start from line element:
  
  \[ c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left( dr^2 + S_k^2(r) d\psi^2 \right) \]

• Angular sizes:
  \[ dl_\perp = R(z) S_k [r(z)] d\psi \]

• Volumes:
  \[ dV(z) = R^3(z) S_k^2 [r(z)] dr d\psi \]
Observational Cosmology

• Angular diameter distance:
  \[ D_A(z) = R(z)S_k[r(z)] \]

• Einstein-de Sitter universe
  \[ D_A(z) = \frac{2c}{H_0(1+z)} \left(1 - (1+z)^{-1/2}\right) \]

• de Sitter universe
  \[ D_A(z) = \frac{cz}{H_0(1+z)} \]
Observational Cosmology

- Angular size:

\[ d\psi(z) = \frac{dl_\perp}{D_A(z)} \]

- Einstein-de Sitter universe

\[ D_A(z) = \frac{2c}{H_0(1+z)} \left( 1 - (1+z)^{-1/2} \right) \]

- de Sitter universe

\[ D_A(z) = \frac{cz}{H_0(1+z)} \]
Observational Cosmology

• Luminosity and flux density:
  • Euclidean space:
    \[ S_v = \frac{L_v(v)}{4\pi r^2} \]
  • Curved, expanding space:
    • \( L = E/t \sim (1+z)^{-2} \)
    • \( L_v = dL/dv \quad d/dv_0 = (1+z)d/dv \)
    • \( v = (1+z)v_0 \)

\[
S_v(v_0) = \frac{L_v[(1+z)v_0]}{4\pi R_0^2 S_k^2 r_z(1+z)}
\]

\[
S_{TOT} = \int \frac{dv}{1+z} S_v = \frac{L_{TOT}}{4\pi R_0^2 S_k^2 (1+z)^2}
\]
Observational Cosmology

- Surface brightness, $I_v$:

\[
\frac{I_v}{v^3} = \frac{1}{(e^{hv/kT} - 1)} = n(v/T) = \frac{B_v[(1+z)v]}{(1+z)^3 v^3}
\]

So the high-redshift objects are heavily dimmed by expansion.
Observational Cosmology

- **Luminosity distance:**
  \[ S_{TOT} = \frac{L_{TOT}}{4\pi D_L^2(z)}, \quad D_L(z) = (1 + z)R_0S_k[r(z)] \]

- **Einstein-de Sitter:**
  \[ D_L(z) = \frac{2c(1 + z)}{H_0} \left(1 - (1 + z)^{-1/2}\right) \]

- **de Sitter:**
  \[ D_L(z) = \frac{cz}{H_0} (1 + z) \]
Observational Cosmology

• Magnitude-redshift relation:

\[ m = M + 5 \log \left( \frac{D_L(z)}{10 \text{pc}} \right) + K(z) \]

• The K-correction: redshift shifts frequency & passbands.

\[ K(z) = -2.5 \log \left( \frac{(1+z)L_v[(1+z)v]}{L_v[v]} \right) \]
Observational Cosmology

• Galaxy Counts: Number of galaxies on sky as function of flux, $N(>S)$.

• Euclidean Model: Consider $n$ galaxies per $\text{Mpc}^3$ with same luminosity, $L$, in a sphere of radius $D$.

\[
D \propto S^{-1/2} \\
V \propto D^3 \propto S^{-3/2} \\
N(> S) = nV(S) \propto S^{-3/2}
\]
Observational Cosmology

• Olbers Paradox:
  The Sky brightness:

\[
I = \frac{1}{A} \int_0^\infty dS \frac{dN}{dS} (> S) S \\
\propto \int_0^\infty dSS^{-3/2} = [S^{-1/2}]_0^\infty
\]

which diverges as S goes to zero.

Too many sources as D increases, due to increase in volume.
Observational Cosmology

• Olbers Paradox: Why is the night sky dark?

\[
I = \frac{1}{A} \int_0^\infty dS \frac{dN(> S)}{dS} S \\
\propto \int_0^\infty dSS^{-3/2} = \left[ S^{-1/2} \right]_0^\infty
\]

which diverges as \( S \) goes to zero.

Too many sources as \( D \) increases, due to increase in volume.
Observational Cosmology

- Relativistic Galaxy Count Model:
  \( \Omega = 1, \ k=0 \) Einstein-de Sitter model:

  \[
  r = 2(1-(1+z)^{-1/2})
  \]

  \[
  V = \frac{A}{3}[R_0 r(z)]^3
  \]

Flux density: \( L_v \sim v^{-\alpha} \)

Number counts.

\( z << 1 \)

\( N(>S) \propto V \propto z^3 \propto S^{-3/2} \)

\( z >> 1 \)

\( N(>S) \propto V \propto \left(\frac{2c}{H_0}\right)^3 \)
Observational Cosmology

- Relativistic Galaxy Count Model:

\[ \log N(>S) \]

Counts converge due to finite volume/age /distance at high-z. So solves Olber’s paradox.
Distances and age of the Universe

• Cosmological Distances: $c/H_0 = 3000h^{-1}\text{Mpc}$.
• Cosmological Time: $1/H_0 = 14 \text{ Gyrs}$.

• Recall our solution for age of Universe:

$$
t_0 = \int_0^\infty \frac{dz}{(1 + z)H(z)}
$$

$$
\approx \frac{2}{3H_0} \left(0.7\Omega_m + 0.3(1 - \Omega_v)\right)^{-0.3}
$$

So if we know $\Omega_m$, $\Omega_v$ and $H_0$, we can get $t_0$. Or if we know $H_0$ and $t_0$ we can get $\Omega_m$ and $\Omega_v$. 
Distances and age of the Universe

• Estimating the age of the Universe, $t_0$:

  • Nuclear Cosmo-chronology:
    Natural clock of radioactive decay, $\tau \sim 10\text{Gyrs}$.

  • Heavy elements ejected from supernova into ISM:
    Thorium (232Th) $\Rightarrow$ Lead (208Pb) 20 Gyrs
    Uranium (235U) $\Rightarrow$ Lead (207Pb) 1 Gyr
    Uranium (238U) $\Rightarrow$ Lead (206Pb) 6.5 Gyrs
Distances and age of the Universe

• Estimating the age of the Universe:
  
  • No new nuclei produced after solar system forms, just nuclear decay:

\[
\Delta D = -\Delta P
= P_0 (1 - e^{-t/\tau})
= P(e^{t/\tau} - 1)
\]

• But don’t know \( D_0 \), so how to measure \( \Delta D \)?
Distances and age of the Universe

1. Estimating the age of the Universe:
   - Take ratio with a stable isotope of D, S.
     \[
     \frac{D}{S} = \frac{D_0}{S} + \frac{P}{S} (e^{t/\tau} - 1)
     \]
   - Plot \(\frac{D}{S}\) versus \(\frac{P}{S}\)

Meteorites: \(\tau_{SS} = 4.57(+/-0.04)\) Gyrs
+ Nuclear theory: \(\tau_{MW} = 9.5\) Gyrs
• Age from Stellar evolution: 
  \( t \sim M/L \)

\[ \tau_{GC} = 13-17 \text{ Gyrs.} \]

Recall for EdS \( t_0=9.3\text{Gyrs}! \)
• Local distance:
  • Use Cepheid Variables (cf Hubbles measurement to M31).
  • Mass $M = 3 - 9 M_{\text{sun}}$
  • Moving onto Red Giant Branch

Luminosity $\sim (\text{Period})^{1.3}$
$L \sim 1/D^2$
$D \sim 1/L^{1/2}$

Need to know $D_{\text{LMC}} = 51 \text{kpc} +/-. 6\%$.
From parallax, or SN1987a.
• Larger distances:

• Use supernova Hubble diagram.
• SN Ia, Ib, II.
• SNIa standard candles.

• Nuclear detonation of WD.

• HST Key programme:
  \[ H_0 = 72 \pm 8 \text{km}s^{-1}. \]

(error mainly distance to LMC)
• So now know $H_0$ and $t_0$ so now know $H_0t_0=0.96$.
• What can we infer about $\Omega_m$ and $\Omega_V$?

$\Omega_v + \Omega_m = 1$

0 $\Omega_v = 0$ $\Omega_m$

0 $H_0t_0$ 2/3

• This implies that if $\Omega_V=0$, $\Omega_m=0$...
• Or $\Omega_V=2.3\Omega_m$! Implies vacuum domination…
• And if flat (k=0) $\Omega_V=0.7$, $\Omega_m=0.3$. 
• Can measure $\Omega_m$ and $\Omega_v$ from luminosity distances to standard candles – the supernova Hubble diagram.

$$D_L(z) = (1 + z) \int_0^z \frac{dz}{H(z)} \approx \frac{c}{H_0} \left( z + (2 + \Omega_m - 2 \Omega_v) z^2 / 4 \right)$$
Cosmological Geometry

- The supernova Type Ia are fainter than expected given their redshift velocity.

Faintness = - log $D_L$

Type 1a supernova Hubble Law

Calan/Tololo (Hamuy et al., A.J. 1996)

Redshift

Accelerating Universe

Decelerating Universe
Can measure $\Omega_m$ and $\Omega_V$ from luminosity distances to standard candles – the supernova Hubble diagram.
Lecture 9
The thermal history of the Universe

- Recall that as Universe expands:
  \[ \rho_m = \rho_{0m} \left( \frac{R}{R_0} \right)^{-3} \]
  \[ \rho_r = \rho_{0r} \left( \frac{R}{R_0} \right)^{-4} \]
  \[ \rho = \rho_{v0} \left( \frac{R}{R_0} \right)^0 \]

- At early enough times we have radiation-dominated Universe.

For \( T(\text{CMB}) = 2.73\,\text{K} \) today.
The thermal history of the Universe

• Also expect a neutrino background, $\rho_v = 0.68 \rho_\gamma$ (see later).

\[ \rho_r = \rho_\gamma + \rho_v \]
\[ \rho_r = a T^4 \]
\[ \Omega_r = 4.2 \times 10^{-3} h^{-2} \]

For $T(\text{CMB}) = 2.73K$ today.

\[ \frac{\rho_m}{\rho_r} = \frac{\rho_{m,0} (1+z)^3}{\rho_{r,0} (1+z)^4} = 1 \]

\[ 1 + z_{eq} = \frac{\rho_{m,0}}{\rho_{r,0}} = \frac{\Omega_{m,0}}{\Omega_{r,0}} = 23,900 (\Omega_m h^2) (T/2.73K)^{-4} \]
The thermal history of the Universe

• How far back do we think we can go to in time?
• To the Quantum Gravity Limit:
  • Quantum Mechanics: de Broglie:
  • General Relativity: Schwarzschild:

\[ \lambda_{dB} = \frac{2\pi c}{v} = \frac{2\pi \hbar}{mc} \]

\[ \lambda_S = \frac{2Gm}{c^2} \]

\[ m_{pl} = \sqrt{\frac{\hbar c}{G}} = 10^{19} \text{ GeV} \]

\[ \ell_{pl} = \sqrt{\frac{\hbar G}{c^3}} = 10^{-35} \text{ m} \]

\[ t_{pl} = \sqrt{\frac{\hbar G}{c^5}} = 10^{-45} \text{ s} \]
Thermal backgrounds

• If expansion rate < interaction rate we have **thermal equilibrium**.
• Shall also assume a we have **perfect gas**.
• Occupation number for relativistic quantum states is:

\[
f(p) = \frac{1}{e^{[\varepsilon(p) - \mu]/kT} \pm 1}
\]

- fermions
- bosons

\[
\varepsilon = \sqrt{m^2 c^4 + p^2 c^2}
\]

\[
e^{-(\varepsilon - \mu)/kT} \quad \text{– Boltzmann}
\]
Thermal backgrounds

• The Chemical Potential, \( \mu \):

\[
dE = TdS - pdV + \mu dN
\]

• A change of energy when change in number of particles.
• As in equilibrium, expect total energy does not change:

\[
\mu = 0
\]
Thermal backgrounds

- The particle number density:
  \[ n = \frac{1}{V} \int dN(p) f(p) \]

- \( N(p) \) is the density of discrete quantum states in a box of volume \( V \) with momentum \( p \):
  \[ p = \hbar k \]

\[ dN(k) = g \frac{V}{(2\pi)^3} d^3k \]

\( g = \) degeneracy factor (e.g., spin states)
Thermal backgrounds

- The number density of relativistic quantum particles:

\[
 n = \frac{g}{\hbar^3} \int \frac{d^3p}{(2\pi)^3} f(p) = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{\varepsilon(p)/kT} \pm 1}
\]

- \(g\) = degeneracy factor (e.g., spin states)
Thermal backgrounds

• The ultra-relativistic limit: $p \gg mc$, $kT \gg mc^2$ (bosons)

$$n = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{p^2/mc^2/kT} - 1} = \frac{\zeta(3)}{\pi^2} g \left( \frac{kT}{\hbar} \right)^3$$

Particle production

• The non-relativistic limit: $kT \ll mc^2$ (Boltzmann)

$$n = \frac{g}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 dp e^{-mc^2/p^2/c^2/kT} = g \left( \frac{mkT}{2\pi\hbar^2} \right)^{3/2} e^{-mc^2/kT}$$

Particle annihilation

![Graph showing Log n vs. Log kT with a threshold at $mc^2$ and a $T^3$ behavior at high temperatures.](image)
Thermal backgrounds

• Proton-antiproton production and annihilation:

\[ \gamma \rightarrow p + \bar{p} \]

• \( m_p = 10^3 \text{ MeV} \) so for \( T > 10^{13} \text{ K} \) there is a thermal background of protons and antiprotons.

• But when \( T < 10^{13} \text{ K} \) annihilation to photons.

• Should annihilate to zero, but in fact \( \Delta p/p = 10^{-9}! \)
  (or else we wouldn’t be here.)

• So there must have been a Matter-Antimatter Asymmetry!!
Thermal backgrounds

- The energy density of relativistic quantum particles (bosons):

\[
u = \rho c^2 = \frac{g}{\hbar^3} \int_0^\infty \frac{d^3p}{(2\pi)^3} f(p)\epsilon(p)
= \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{\epsilon(p)/kT} - 1} \epsilon(p)
= \frac{g\pi^2}{30\hbar^3 c^3} (kT)^4
\]

\(g = \) degeneracy factor (eg spin states)
Thermal backgrounds

• The entropy, $S$, of relativistic quantum particles:
  • The entropy is an extensive quantity (like $E$ & $V$):

\[
\begin{align*}
E_1 & \quad V_1 \quad S_1 \\
E_2 & \quad V_2 \quad S_2
\end{align*}
\]

\[E = E_1 + E_2 \quad V = V_1 + V_2 \quad S = S_1 + S_2\]

So

\[\frac{dE(V,T)}{dV} = T dS(V,T) - p dV\]

\[
\frac{E}{V} dV + \frac{\partial E}{\partial T} dT = \left( T \frac{S}{V} dV + T \frac{\partial S}{\partial T} dT \right) - p dV
\]

\[
\Rightarrow \quad \frac{E}{V} = \frac{TS}{V} - p
\]

Hence

\[
S = \frac{S}{V} = \frac{\rho + p / c^3}{T}
\]
Thermal backgrounds

- So in the ultra-relativistic case:

\[ n \propto T^3 \]
\[ u \propto T^4 \]
\[ p = \frac{1}{3} u \]
\[ \Rightarrow s = \frac{4}{3} \frac{\rho}{T} \propto T^3 \]

- So \( s \propto n \)

- But \( \dot{s} = 0 \) (entropy is a conserved quantity).

- Usual to quote ratios e.g. baryon density \( n_B/s = 10^{-9} \).
Thermal backgrounds

• Given these simple scalings with $T$ for bosons, what is the scaling for $n$, $u$ and $s$ for fermions when $kT >> mc^2$?

• Formally expand:

\[
\frac{1}{e^x + 1} = \frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1}
\]

• So occupation numbers:

\[
f_F(T) = f_B(T) - 2f_B(T/2)
\]
Thermal backgrounds

• Given these simple scalings with $T$ for bosons, what is the scaling for $n$, $u$ and $s$ for fermions when $kT \gg mc^2$?

• For $kT\gg mc^2$:

\[
\begin{align*}
 n_F &\propto g_F T^3 \\
n_B &\propto g_B T^3
\end{align*}
\]

• Number densities:

\[
n_F(T) = \frac{g_F}{g_B} \left[n_B(T) - 2n_B(T/2)\right]
= \frac{g_F}{g_B} n_B(T) \left[1 - \frac{2}{2^3}\right]
= \frac{3}{4} \frac{g_F}{g_B} n_B(T)
\]
Thermal backgrounds

• Given these simple scalings with $T$ for bosons, what is the scaling for $n$, $u$ and $s$ for fermions when $kT >> mc^2$?

• For $kT >> mc^2$:

\[ u_F \propto g_F T^4 \]
\[ u_B \propto g_B T^4 \]

• Energy densities:

\[
\begin{align*}
    u_F(T) &= \frac{g_F}{g_B} \left[ u_B(T) - 2u_B(T/2) \right] \\
    &= \frac{g_F}{g_B} u_B(T) \left[ 1 - \frac{2}{2^4} \right] \\
    &= \frac{7}{8} \frac{g_F}{g_B} u_B(T)
\end{align*}
\]
Thermal backgrounds

• Given these simple scalings with $T$ for bosons, what is the scaling for $n$, $u$ and $s$ for fermions when $kT \gg m_c^2$?

• For $kT \gg m_c^2$:

\[ s = \frac{u + p}{c^2 T}, \quad u_F \propto g_F T^4, \quad p = u / 3 \]

\[ u_B \propto g_B T^4 \]

• Energy densities:

\[ s_F(T) = \frac{g_F}{g_B c^2 T} [u_B(T)(1+1/3) - 2u_B(T/2)(1+1/3)] \]

\[ = \frac{g_F u_B(T)}{g_B c^2 T} \frac{4}{3} \left[ 1 - \frac{2}{2^4} \right] \]

\[ = \frac{7}{8} \frac{g_F}{g_B} s_B(T) \]
Thermal backgrounds

- Given these simple scalings with $T$ for bosons, what is the scaling for $n$, $u$ and $s$ for fermions when $kT \gg mc^2$?

\[
\begin{align*}
 n_F(T) &= \frac{3}{4} \frac{g_F}{g_B} n_B(T), \\
 u_F(T) &= \frac{7}{8} \frac{g_F}{g_B} u_B(T), \\
 s_F(T) &= \frac{7}{8} \frac{g_F}{g_B} s_B(T)
\end{align*}
\]

- Define an effective number of relativistic particles:

\[
g^*_\text{Total} = \sum \text{Bosons} \frac{7}{8} \sum \text{Fermions} g_F
\]

- So energy of all relativistic particles is:

\[
u_{Total} = \frac{g^*_\pi}{30 (\hbar c)^3} (kT)^4
\]
Thermal backgrounds

• The effective number of relativistic particles will change with time as $kT < mc^2$ and particles become non-relativistic.

$$g_* = \sum_{\text{Bosons}} g_B + \frac{7}{8} \sum_{\text{Fermions}} g_F$$

• For high-$T$ $g_* = 100$. If supersymmetric, $g_* = 200$. 
Time and Temperature

• At early times radiation and matter are strongly coupled and thermalized to temperature, T, of radiation.

• Recall:

\[ t = \sqrt{\frac{3}{32\pi G \rho_r}} \]  

in a radiation-dominated universe,

and:

\[ \rho_r \propto g_* T^4, \quad \text{so} \quad t \propto g_*^{-1/2} T^{-2} \]

• Hence:

\[ t = g_*^{-1/2} \left( \frac{T}{1.8 \times 10^{10} K} \right)^{-2} \text{ seconds} \]

• Note also that \( T=2.73T(1+z), \) so \( z \sim T/(1 \text{ K}). \)
• With time now related to temperature, and hence energy, we can map out the thermal history of the Universe.
- $z \sim T/(1K)$
- $E = kT \sim T/(3 \times 10^3)eV$

$\text{Dark Matter formed?}$

$\text{Inflation?}$
Freeze-out and Relic Particles

• Electrons - Positrons annihilation:

• For first 3 seconds we have $e^- e^+ \leftrightarrow \gamma \gamma$

• Then $T$ drops and energy in $\gamma$ becomes too low, so electrons and positrons annihilate.

• Stops when annihilation rate drops below expansion rate.
Freeze-out and Relic Particles

- Need the Boltzmann Equation to describe reactions:

\[
\dot{n} + 3Hn = -\langle \sigma v \rangle n^2
\]

Rate of change

- Timescales:

\[
\tau_{\text{exp}} \sim 1/H \sim R^2
\]

Dilution by expansion

\[
\tau_{\text{int}} \sim 1/(\langle \sigma v \rangle n) \sim R^3
\]

Loss due to annihilation

\[
\log \tau > \log R
\]

thermal equilibrium

\[
\log \tau < \log R
\]

Particle Freeze-out/ Decoupling
Freeze-out and Relic Particles

- Creation, annihilation and freeze-out of particle relics:

$$T^3 n^{-1/3} \log n$$

Pair production

Freeze-out

Pair annihilation

$$T^{3/2} e^{-m c^2 / kT}$$

Log n

T^3

log kT
Freeze-out and Relic Particles

• Electrons-Positrons annihilation and neutrino decoupling:

• What happens to the energy released by $e^- e^+ \rightarrow \gamma \gamma$?

• At early times only have photons, neutrinos and $e^- e^+$ pairs in equilibrium.

• At $T = 5 \times 10^9 K$ (3 seconds) $e^- e^+$ pairs annihilate.

As weak force decoupled at $T=10^{10} K$. 

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Freeze-out and Relic Particles

• So radiation is boosted above neutrino temperature by neutrino decay.

• Before $n_\nu \sim n_\gamma$, after $n_\nu < n_\gamma$ and $T_\nu < T_\gamma$.

• But recall entropy is conserved $\dot{s} = 0$

$$\implies s \propto g^* T^3$$

where

$$g^* = \sum_{\text{Bosons}} g_B + \frac{7}{8} \sum_{\text{Fermions}} g_F$$
Freeze-out and Relic Particles

• How much is photon temperature boosted?

\[ s \propto g * T^3 \]

• Entropy:
  \( g_e = 2 \)
  \( g_\gamma = 2 \)

\[ s_{\text{after}} (\gamma) = s_{\text{before}} (\gamma + e^+ + e^-) \]
\[ = \left( 1 + \frac{7}{8} \left[ \frac{g_{e^+} + g_{e^-}}{g_\gamma} \right] \right) s_{\text{before}} (\gamma) \]
\[ = \frac{11}{4} s_{\text{before}} (\nu) \]

• Neutrino Temperature:

\[ T_v = \left( \frac{4}{11} \right)^{1/3} \]
\[ T_\gamma (= 2 / 73 K) = 1.95 K \]
Freeze-out and Relic Particles

- How much is radiation energy boosted?

- Neutrino energy:

\[ u_v = \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} \]

\[ u_\gamma = 0.227 \, u_\gamma \]

- Enhances \( \rho_r \) by factor of 1.68 for 3 neutrino species:

\[ \rho_r = (u_\gamma + 3u_v) c^2 \]

\[ = 1.68 \, \rho_\gamma \]

![Graph showing the relationship between \( \rho_r \) and \( \rho_\gamma \) with \( \rho_v \) and \( \rho_\gamma \) as markers. Neutrinos annihilate.]
Relic massive neutrinos

• Can we put cosmological constraints on the mass of neutrinos?

• 1960’s: Particle physics models with $m_\nu = 0$.

• 1970’s: Particle physics models with massive neutrinos.

• 1990’s: Non-zero mass detected (Superkamiokande, and Sudbury Neutrino Observatory (SNO) confirms Solar model).
Relic massive neutrinos

• Can we put cosmological constraints on the mass of neutrinos?

• Number-density of cosmological neutrinos:

\[ n_F (\nu + \bar{\nu}) = \frac{3}{4} n_B (\gamma | T_\gamma = 1.95 \, K) \]

\[ = 113 \, \nu / cm^3 / species \]

• Mass-density:

\[ \rho_\nu = m_\nu n_\nu = \frac{3\Omega_\nu H^2}{8\pi G} \]
Relic massive neutrinos

• Can we put cosmological constraints on the mass of neutrinos?

• Density-parameter of cosmological neutrinos:

\[ \Omega_\nu = \frac{1}{93.5 \ h^2 \ eV} \sum_{i=1}^{N_\nu} m_{\nu_i} \]

• Re-arrange:

\[ \overline{m}_\nu = \frac{1}{3} \sum_{i=1}^{N_\nu} m_{\nu_i} \leq 4.6 eV \left( \frac{N_\nu}{3} \right)^{-1} \left( \frac{\Omega_\nu}{0.3} \right) \left( \frac{h}{0.7} \right)^2 \]

• Compare with lab: \( m_{\nu_e} < 15 \text{eV}, m_{\nu_\mu} < 0.17 \text{MeV}, m_{\nu_\tau} < 24 \text{MeV} \).

\( \Delta m^2 = m_i^2 - m_j^2 = [(7 \times 10^{-3} \text{eV})]^2 \).
Big-Bang Nucleosynthesis (BBN)

• As $R$ tends to zero and $T$ increases, eventually reach nuclear burning temperatures.

• 1940’s: George Gamow suggests nuclear reactions in early Universe led to Helium.
  • Prediction of a radiation background (CMB)
  • Predicted 25% helium by mass, as found in stars.

• 1960’s: Details worked out by Hoyle, Burbidge and Fowler.

• First need free protons and neutrons to form.
Big-Bang Nucleosynthesis (BBN)

So first need proton & neutron freeze-out.

Recall:

- Below $T < 10^{13}$K ($M_p \sim M_n \sim 10^3$ MeV):
  
  $p\bar{p} \rightarrow \gamma\gamma$
  
  $n\bar{n} \rightarrow \gamma\gamma$

Annihilation leaves a residual $\Delta p/p \sim \Delta n/n \sim 10^{-9}$.

Protons and neutrons undergo Weak Interactions:

$p + e^- \leftrightarrow n + \nu$

$n + e^+ \leftrightarrow p + \bar{\nu}$
**Big-Bang Nucleosynthesis (BBN)**

- Assume equilibrium and low energy ($kT \ll mc^2$) limit:

$$n \propto T^{3/2} e^{-mc^2 / kT}$$

- Ratio of neutrons to protons at temp $T$ ($\Delta m = m_n - m_p = 1.3\text{ MeV}$):

$$\frac{n_n}{n_p} = e^{-\Delta mc^2 / kT} \approx e^{-1.5 \times 10^{10} K / T}$$
Big-Bang Nucleosynthesis (BBN)

- Annihilations stop when $p$ & $n$ freeze-out occurs:

$$\frac{\tau_{\text{int}}}{\tau_{\text{exp}}} \approx \frac{1}{\sigma v} \approx \frac{1}{H}$$

- So ratio is frozen in at $T_{\text{freezeout}}$

$$\frac{n_n}{n_p} = e^{-\frac{\Delta mc^2}{kT_{\text{freezeout}}}} \approx e^{-1.5 \times 10^{10} \frac{K}{T_{\text{freezeout}}}}$$
Big-Bang Nucleosynthesis (BBN)

- What is the observed neutron-proton ratio?

- Most He is in the form of $^4$He:

- He fraction by mass is:
  \[
  Y = \frac{(4 \times n_n / 2)m}{(n_n + n_p)m} = \frac{2}{1 + n_p / n_n}
  \]

- Observe $Y=0.25$ for stars.

- So $n_p/n_n=2/Y-1=7$, or:
  \[
  \frac{n_n}{n_p} \approx \frac{1}{7} \approx 0.14
  \]
Big-Bang Nucleosynthesis (BBN)

• At what time, then, does neutron freeze-out happen?

• Need to know weak interaction rates: $<\sigma v>_{\text{weak}}$

• This was calculated by Enrico Fermi in 1930’s.

• Find

$$T_{\text{freezeout}}(n) \approx 1.4 \times 10^{10} \, K$$

• So expected neutron-proton ratio is:

$$\frac{n_n}{n_p} \approx e^{-1.5 \times 10^{10} K / T_{\text{freezeout}}} \approx 0.34$$
Big-Bang Nucleosynthesis (BBN)

- Expect $n_n/n_p = 0.34$.
- But we said $\frac{n_n}{n_p} = 0.14$ from observed stellar abundances.
- Close, but a bit big.
- But:
  1. We have assumed $kT_{\text{freezeout}} \gg m_e c^2$ but really $kT_{\text{freezeout}} \sim m_e c^2$
  2. Neutrons decay. $\tau_n = 887 \pm 2$ seconds for free neutrons. Need to be locked away in a few seconds:

$$\frac{n_n}{n_p} \approx 0.34 \rightarrow 0.14 \quad \Rightarrow \quad Y = 0.25$$
Big-Bang Nucleosynthesis (BBN)

• The onset of Nuclear Reactions.

• At the same time nuclear reactions become important.

• Neutrons get locked up in **Deuteron** via the strong interaction

\[ n + p \rightarrow D + \gamma \]

• Happens at deuteron binding energy \( kT \approx 2.2 \text{MeV} \)

• Dominant when \( T(\text{D formation}) = 8 \times 10^8 \text{K} \), or at a time \( t = 3 \) minutes.
Big-Bang Nucleosynthesis (BBN)

- The formation of Helium.

- $^4\text{He}$ is preferred over H or D on thermodynamic grounds. Binding energies: $E(\text{He}) = 7 \text{ MeV}$ $E(\text{D}) = 1.1 \text{ MeV}$

- After Deuteron forms:

  \[ D + D \rightarrow ^3\text{He} + n \]
  \[ D + D \rightarrow T + p \]

- Then

  \[ D + D \rightarrow ^4\text{He} \]
  \[ T + p \rightarrow ^4\text{He} \]
  \[ D + ^3\text{He} \rightarrow ^4\text{He} + p \]
  \[ T + D \rightarrow ^4\text{He} + n \]

  T gets too low and reactions stop at Li & Be.

BBN starts at $10^{10} \text{K}$, $t=1\text{s}$. Ends at $10^9 \text{K}$, $t=3\text{mins}$. 

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Big-Bang Nucleosynthesis (BBN)

- **Summary of BBN:**

  - $T=10^{13}\text{K}, t\sim0.1\text{s}$: Neutron & proton annihilation ($\Delta p/p\sim\Delta n/n\sim10^{-9}$).

  - $T=10^{10}\text{K}, t=1\text{s}$: Neutron freeze-out ($n_n/n_p=0.34.$) Neutrons decay ($n_n/n_p=0.14.$) Nuclear reactions start.

- $n + p \rightarrow D + \gamma$

  - $T=10^9\text{K}, t=3\text{mins}$: Formation of Helium. Peak of D formation. End of nuclear reactions $\text{H, D, } ^3\text{He, } ^4\text{He, } ^7\text{Li, } ^7\text{Be}$
Big-Bang Nucleosynthesis (BBN)

- The number of neutrino generations:

- The ratio of neutrons to protons is:

\[
\frac{n_n}{n_p} = f_{\text{Decay}} e^{-\frac{\Delta mc^2}{kT_{\text{freezeout}}}} \approx 0.163 (\Omega_B h^2)^{0.04} \left( \frac{N_\nu}{3} \right)^{0.2} \approx 0.14
\]

- Depends weakly on \( \rho_B \): higher baryon density means closer packed, so n locked up in nucleons (D) faster.

- Depends on \( N_\nu \): More neutrinos, more \( \rho_\tau (g_*) \), so Hubble rate increases, neutron freezeout happens sooner, so more n.

- Cosmological constraint that \( N_\nu < 4 \).

- In 1990’s LEP at CERN sets \( N_\nu = 3 \).
Big-Bang Nucleosynthesis (BBN)

• Testing BBN:
  • The abundance of elements is sensitive to density of baryons:
    • $\eta$ is number density of baryons per unit entropy.

$$\eta = \frac{n_n + n_p}{n_\gamma} \approx 3 \times 10^{-10}$$

• This gives us the matter-antimatter difference of $\Delta p/p \sim 10^{-9}$

Agreement between BBN theory and observation is a spectacular confirmation of the Big-Bang model!
Big-Bang Nucleosynthesis (BBN)

• Using BBN to weigh the baryons:

• The abundance of elements is sensitive to the density of baryons.

• The photon density scales as $T^{-3}$, so:

$$\eta = \frac{n_n + n_p}{n_\gamma} = 2.74 \times 10^{-8} (\Omega_B h^2) \left( \frac{T}{2.73K} \right)^{-3} \approx 3 \times 10^{-10}$$

• This yields

$$\Omega_B h^2 = 0.02 \pm 0.002 \quad \Rightarrow \quad \Omega_B = 0.04 \left( \frac{h}{0.7} \right)^{-2}$$

• But:

$$\Omega_m \approx 0.3 >> \Omega_B$$

So most of the matter in the Universe cannot be made of Baryons!
Recombination of the Universe

- Energy-density of the Universe:

$$\log \rho$$

Plasma Era:
\(\gamma, p, e^-\) in a plasma
Thomson scattering (g-e)

Recombination (a misnomer)

Bound atoms (H) and free photons

T = 10^3K

\(z_{\text{rec}} \approx 10^3\)

\(z_{\text{eq}} \approx 10^4\)
Recombination of the Universe

• Ionization of a plasma:

• Assume thermal equilibrium.

• Use Saha equation for ionization fraction, $x$:

$$x = \frac{H^+}{H^+ + H}$$

$$\frac{x^2}{1-x} = \frac{(2\pi m kT)^{3/2}}{n (2\pi \hbar)^3} e^{-x/kT}$$

$\chi = 13.6$ eV – H binding energy.
Recombination of the Universe

- Ionization of a plasma:
  - But equilibrium rapidly ceases to be valid. Interactions are too fast, and photons cannot escape.

\[ H \quad E = \hbar \omega >> \chi \quad \infty \]

\[ H^+ \quad \]

- Escape bottleneck with 2-photon emission:

\[ H^+ \quad 2s \quad E = \hbar \omega < \chi \]

\[ H \quad 1s \quad \gamma \]

- This means the ionization fraction is higher than predicted by the Saha equation.
Recombination of the Universe

• The surface of last scattering:
  • In plasma photons random walk (Thomson scattering off electrons).
  • After recombination photons travel freely and atoms form.

The last scattering surface forms a photosphere (like sun), The Cosmic Microwave Background.
The Cosmic Microwave Background

• **The CMB spectrum:**

• The CMB was discovered by accident in 1965 by Arno Penzias and Bob Wilson, two researchers at Bell Labs, New Jersey.

• This confirmed the Big-Bang model, and ruled out the competing Steady-State model of Hoyle.

• They received the 1978 Nobel Prize for Physics.
The Cosmic Microwave Background

• The CMB spectrum:
  • The Big Bang model predicts a thermal black-body spectrum (thermalized early on and adiabatic expansion).
  • The observed CMB is an almost perfect BB spectrum:

\[ T = 2.725 \pm 0.002 \, K \]

• Accuracy limited by reference BB source.

• CMB contributes to 1% of TV noise.
• The CMB dipole:
  • The CMB dipole is due to our motion through the universe.

• Doppler Shift: \( v = D v_0 \), \( D = 1 + v \cdot r / rc \) - dipole.

\[
n_\gamma (v) = \left( e^{hv/kT} - 1 \right)^{-1} \rightarrow \left( e^{hv/DkT_0} - 1 \right)^{-1}
\]

• Same as temperature shift: \( T = (1 + v \cdot r / rc) T_0 \)
The Cosmic Microwave Background

• The CMB dipole:
  • This gives us the absolute motion of the Earth (measured by George Smoot in 1977):
    \[ V_{\text{Earth}} = 371 \pm 1 \text{ km/s,} \]
    \[ (l,b) = (264^o,48^o) \]
    assuming no intrinsic dipole.

  • What is its origin?
    • Not due to rotation of sun around galaxy (wrong direction).
      \[ v=300\text{km/s}, \ (l,b)=(90^o,0^o). \]
    • Motion of the Local Group?
      • Implies \( V_{\text{LG}}=600\text{km/s} \ (l,b)=(270^o,30^o). \)
Lecture 13
The Cosmic Microwave Background

• The CMB dipole:
  • What is its origin of the dipole?
    • Motion of the Local Group.
    • $V_{LG} = 600 \text{km/s} \ (l,b) = (270^\circ, 30^\circ)$.

• Motion due to gravitational attraction of large-scale structure:
  • LG is falling into the Virgo Supercluster (~10Mpcs away)
  • Which is being pulled by the Hercules Supercluster (the Great Attractor, ~150Mpc away).
Dark Matter

• Recall from globular cluster ages, supernova and BBN:

\[ \Omega_m \approx 0.3 \gg \Omega_B = 0.04 \]

• So we infer most of the matter in the Universe is non-baryonic.

• How secure is the density parameter measurement?

• If it’s wrong and lower, could all just be baryons.
Dark Matter

• Mass-to-light ratio of galaxies:

\[ \rho_m = \left( \frac{M}{L} \right) \rho_L \]

• We can expect \( M/L = F(M) \)

Comets:
\[ \frac{M}{L} \approx 10^{12} \frac{M_{\text{Sun}}}{L_{\text{Sun}}} \]

Low-Mass stars:
\[ \frac{M}{L} \approx 10 \frac{M_{\text{Sun}}}{L_{\text{Sun}}} \]

Galaxy stars:
\[ \frac{M}{L} \approx 1 - 10 \frac{M_{\text{Sun}}}{L_{\text{Sun}}} \]
Dark Matter

• In blue star-light:

\[ \rho_L = (2.0 \pm 0.7) \times 10^8 h L_{\text{Sun}} \, Mpc^{-3} \]
\[ \rho_{\text{crit}} = 2.78 \times 10^{11} \Omega_m h^2 M_{\text{Sun}} \, Mpc^{-3} \]

• So we find:

\[ \left( \frac{M}{L} \right)_{\text{Blue}} \approx (300 \pm 100) \left( \frac{\Omega_m}{0.3} \right) \left( \frac{h}{0.7} \right) \frac{M_{\text{Sun}}}{L_{\text{Sun}}} \]

This is way above the M/L=10 we see in stars.

• So not enough luminous baryons in stars.

• In fact not enough baryons in stars to make \( \Omega_B = 0.04 \), so there must be baryonic dark matter too.
Dark Matter

- Dark Matter in Galaxy Halos:
  - In 1970’s Vera Rubin found galaxies rotate like solid spheres, not Keplerian.

- For $V =$ const, need $M(<r) \sim r$, so

- Density profile of Isothermal Sphere.
- Yields dark matter $= 5 \times$ stellar mass
• In 1933 Fritz Zwicky found the Doppler motion of galaxies in the Coma cluster were moving too fast to be gravitationally bound.
• First detection of dark matter.

Zwicky
(1898-1974)

• Assume hydrostatic equilibrium:

\[ F = -\frac{GM(<r)}{r^2} = \frac{1}{\rho} \frac{\partial}{\partial r} P = \frac{1}{\rho} \frac{\partial}{\partial r} \rho \sigma_v^2 \]

• So need 10 – 100 x stellar mass.
Dark Matter in Galaxy Clusters

- X-ray emission from galaxy clusters.
- Hot gas emits X-rays.
- Assume hydrostatic equilibrium.
- Equate gravitational and thermal potentials:

\[
\frac{GM(<r)}{r} = -\frac{kT(r)}{\mu m_p} \left( \frac{d \ln T}{d \ln r} + \frac{d \ln \rho_{gas}}{d \ln r} \right)
\]

- Get both total mass, and baryonic (gas) mass.

\[
\frac{M_B}{M_{Tot}} = \frac{M_{gas} + M_{stars}}{M_{gas} + M_{stars} + M_{DM}} = 0.01 + 0.09 \left( \frac{h}{0.7} \right)^{-3/2} \approx 0.1
\]

- So \( M_{DM} = 10 M_B \)
Dark Matter in Galaxy Clusters

- Gravitational lensing by clusters of galaxies.
- Use giant arcs around clusters to measure projected mass.
- Strongest distortion at the Einstein radius:
  \[ \theta_E \propto \sqrt{M(\leq \theta_E)} \]
- Independent of state of cluster (equilibrium).
- Find again \( M_{\text{Tot}} = 10 - 100 M_{\text{stars}} \)
Dark Matter and $\Omega_m$

- So independent methods show in galaxy clusters:
  \[ M_{DM} \approx 10 M_{\text{gas}} \approx 100 M_{\text{stars}} \]

- Can estimate mass-density of Universe from clusters:
  \[
  \rho_m = M_{\text{cluster}} n_{\text{cluster}} \approx 10^{14} M_{\odot} 10^{-3} \text{Mpc}^{-3}
  \]
  \[
  \rho_{\text{crit}} = 2.78 \times 10^{11} M_{\odot} \text{Mpc}^{-3}
  \]
  \[
  \Omega_m \approx 0.3
  \]
The distribution of matter in the Universe is not uniform.

There exists galaxies, stars, planets, complex life etc.

Where does all this structure come from?

Is there a fossil remnant from when it was formed?

How do we reconcile this structure with the Cosmological Principle & Friedmann model?
The large-scale distribution of galaxies

6 billion light years

The 2-degree Field Galaxy Redshift Survey (2dFGRS)
The large-scale distribution of galaxies

The 2-degree Field Galaxy Redshift Survey (2dFGRS)
Large-scale structure in the Universe

• The matter density perturbation:

\[ \delta(r) = \frac{\rho(r) - \langle \rho \rangle}{\langle \rho \rangle} \]

• Fourier decomposition:

\[ \delta(r) = \sum_k \delta_k e^{-ik \cdot r} \]

\[ k_x = n \pi / L, \quad n = 1, 2, \ldots \]
Large-scale structure in the Universe

• The statistical properties:

  – The Ergodic Theorem:

    $$\langle \ldots \rangle = \frac{1}{V} \int d^3 r \ldots$$

    Volume averages are equal to ensemble averages.

• Moments of the density field:

  $$\langle \delta \rangle = 0, \quad \langle \delta^2 \rangle = \frac{1}{V} \int d^3 r \, \delta^2(r) = \frac{1}{V} \int d^3 r \left| \sum_k \delta_k e^{-i k \cdot r} \right|^2 = \sum_k |\delta_k|^2$$

• Define the power spectrum:

  $$P(k) = |\delta_k|^2$$

  $$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} = \frac{d\langle \delta^2 \rangle}{d \ln k} \approx \langle \delta^2 \rangle (R \approx 2\pi / k)$$
Large-scale structure in the Universe

- The statistical properties:
- The correlation function:

\[ \xi(r) = \langle \delta(x)\delta(x+r) \rangle = \frac{1}{V} \int d^3x \, \delta(x)\delta(x+r) = \frac{1}{V} \int d^3x \sum_k \delta_k e^{-ikr} \sum_{k'} \delta_{k'} e^{-ik'(x+r)} = \sum_k |\delta_k|^2 e^{-ikr} \]

- So correlation function is the Fourier transform of the power spectrum, \( P(k) \).

- For point processes, correlation function is the excess probability of finding a point at 2 given a point at 1:

\[ dP(2 \mid 1) = n^2 (1 + \xi(r)) dV_1 dV_2 \]
Lecture 14
Large-scale structure in the Universe

• The Matter Power Spectrum:
  • So 2-point statistics can be found from $P(k)$.
  • What is the form of $P(k)$?
  • For simplicity let’s assume for now it’s a power-law:

$$P(k) = A k^n$$
$$\Delta^2(k) = \frac{A k^{n+3}}{2\pi^2}$$

where $A$ is an amplitude and $n$ is the spectral index.
Large-scale structure in the Universe

- The Potential Power Spectrum:
  - Can we put limits on spectral index, $n$?
  - Consider the potential field, $\Phi$.
  - So far we have assumed $\Phi \ll 1$ (so metric is Friedmann).
  - Poisson equation:

$$\nabla^2 \Phi = 4\pi G \rho_0 \delta$$

$$- k^2 \Phi_k = 4\pi G \rho_0 \delta_k$$

$$\Phi_k = -4\pi G \rho_0 \delta_k k^{-2}$$

- So

$$P_\Phi(k) = \left< |\Phi_k|^2 \right> \propto k^{n-4}$$

$$\Delta^2_\Phi(k) \propto k^{n-1}$$
Large-scale structure in the Universe

- The Potential Power Spectrum:
  - Can we put limits on spectral index, n?
  - To keep homogeneity, need n less than or equal to 1.
  - To avoid black holes, need n greater or equal to 1.
  - So must have n=1, with $\Delta^2_\Phi = \text{const} \sim 10^{-10}$ (from CMB).
  - n=1 is scale invariant (fractal) in the potential field.
Structure in the Universe

• Where did this structure come from?
• In 1946 Russian physicist Evgenii Lifshitz suggested small variations in density in the Early Universe grow due to gravitational instability.
Dynamics of structure formation

• Consider the gravitational collapse of a sphere:
• Assume Einstein-de Sitter ($\Omega_m=1$, $p_m=0$).

• Behaves like a mini-universe, so

$$r = A(1 - \cos \theta)$$
$$t = B(\theta - \sin \theta)$$
Dynamics of structure formation

Since \( \dot{r} = -\frac{GM}{r^2} \) we find \( A^3 = GM B^2 \)

- Linear theory growth:

\[
r = A(1 - \cos \theta) \approx A\frac{1}{2} \theta^2 (1 - \frac{1}{12} \theta^2)
\]

\[
t = B(\theta - \sin \theta) \approx B\frac{1}{6} \theta^3 (1 - \frac{1}{20} \theta^2)
\]

- 0th order:

\[
r_0 = \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3} \propto a(t)
\]

- 1st order:

\[
r = \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3} \left( 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right)
\]

- expansion of universe.
Dynamics of structure formation

• Linear growth of over-densities:

\[ r = \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3} \left( 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right) \]

• Density \( \sim 1/\text{Vol} \):

\[ \frac{\rho}{\rho_0} = 1 + \delta = \left( \frac{r}{r_0} \right)^3 \]

\[ r = r_0 (1 + \delta)^{-1/3} \approx r_0 (1 - \frac{1}{3} \delta) \]

• Linear growth in E-dS:

\[ \delta = \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3} \propto a(t) \]
Dynamics of structure formation

- Nonlinear growth of over-densities:
  - Linear growth: $\delta \propto \alpha$
  - Turnaround: $\theta = \pi, \ t = B\pi, \ \delta = 5.55$
  - Collapse: $\theta = 2\pi, \ r = 0, \ \delta = \infty$
  - Virialization: $V = -2K, \ r = \frac{1}{2}r_{\text{max}}, \ \theta = \frac{3}{2\pi}, \ \delta = 177, \ \delta_{\text{lin}} = 1.69$
Formation of a galaxy cluster
Dark Matter and the Power Spectrum

- Since $\delta \sim a$ on all linear scales, the matter power spectrum preserves its shape in the linear regime.

- Linear regime, $\delta \ll 1$, valid at early times and on large scales.
- If $n > -3$ initial shape will be preserved on large scales.

- Power spectra is shaped by dark matter, so leaves imprint.
Dark Matter and the Power Spectrum

• Have already seen we need non-baryonic dark matter
• \( \Omega_B = 0.04 < \Omega_m = 0.3 \), from BBN and clusters, SN, ages…).

• But what can the dark matter be?
  • Massive Neutrinos?
    Now know to have mass, so possibly.
  • Black holes?
    Also now know to exist at centre of all galaxies. But if too large, disrupt galactic disk & lens stars in LMC and galactic bugle (MACHO and OGLE surveys). Too small and will over-produce Hawking radiation emission.
  • A frozen-out particle relic from the early universe:
    Weakly Interacting Massive Particles (WIMPS).
What is Dark Matter?

• Must be weakly interacting to avoid detection so far.
• A promising idea in particle physics is Supersymmetry:

  Matter Particles (fermions)   Force Particles (boson)

  electron   photon
  photino    selectron

• The lightest supersymmetric particle (the neutralino = gravitino+photino+zino) could be detected in 2007 at Europe’s CERN Large Hadron Collider (LHC).
Dark Matter and the Power Spectrum

- It is convenient to divide dark matter candidates into 3 types:
  1. Hot Dark Matter (HDM):
     Relativistic at freeze-out (e.g. neutrinos), $kT \gg mc^2$.
     \[
     n_{HDM} \approx n_\gamma, \quad m_{HDM} \approx eV
     \]
  2. Warm Dark Matter (WDM):
     Some momentum at freeze-out, $kT \sim mc^2$.
     \[
     n_{WDM} < n_\gamma, \quad m_{WDM} \approx 1 - 10 keV
     \]
  3. Cold Dark Matter (CDM):
     No momentum at freeze-out, $kT \ll mc^2$.
     \[
     \rho = mn \propto me^{-m/MeV}
     \]
Dark Matter and the Power Spectrum

• The matter Transfer Functions:
  Dark matter affects the matter power spectrum of density perturbations.

• **HDM: Free-streaming and damping:** HDM freezes-out relativistically.

\[ \log \Delta_k^2 \]

\[ \sim a^2 \]

\[ k_H \sim 1/L_{damp} \]

\[ L_{damp} = \text{Damping scale} \]

\[ n=1 \]

\[ \lambda >> ct \]

\[ \lambda << ct \]

HDM trapped

HDM escapes
Lecture 15
• **HDM: Free-streaming and damping:**
  
  • HDM freezes-out relativistically, $v \sim c$, so can free-stream out of density perturbations in matter-dominated regime.

$$L_{damp} \approx ct(kT = mc^2) \approx D_{HI}(z_{eq}) \propto \left(\Omega_m z_{eq}\right)^{-1/2}, \quad z_{eq} = 23,900(\Omega_m h^2)$$

$$L_{damp}(\text{neutrino}) \approx 16(\Omega_\nu h^2)^{-1} \text{Mpc}$$

• So if HDM, expect no structure (galaxies) on small-scales today!.
• This rules out an HDM-dominated universe.
• **Baryons + photons:** Baryon Oscillations and Silk damping.
  • \( t < t_{\text{rec}} \): Baryon-photon plasma

\[ \log \Delta_k^2 \sim a^2 \]

\[ k_H \sim 1/D_H, \quad D_H = \text{Horizon scale} \]

Photons & baryons trapped

\( \lambda >> ct \)

\( \lambda << ct \)

Photons & baryons trapped in plasma

- No collapse
• **Baryons + photons**: Baryon Oscillations and Silk damping.

• \( t > t_{\text{rec}} \): Baryon and photons free. Baryons oscillate.

\[
\log \Delta_k^2 \quad \sim a^2 \\
\lambda \gg ct \quad \lambda \ll ct
\]

\( k_H \sim 1/D_H \)

\( D_H = \) Horizon scale

\( \lambda = \) Horizon scale

\( n = 1 \)

**Photons free-stream carrying baryons (Silk damping).**

**Baryons oscillate.**
• CDM + photons: The Meszaros Effect.
  • Recall at early times $\rho_\gamma \gg \rho_m$

\[ \log \Delta_k^2 \approx a^4 \]

$\lambda \gg ct$ \hspace{2cm} $\lambda \ll ct$

$D_H = \text{Horizon scale}$

$\kappa_H \approx 1/D_H$
• **CDM + photons: The Meszaros Effect.**
  • After matter-radiation equality, all scales grow the same.
  • Produces a break in the matter power spectrum at comoving horizon scale at $z_{eq} = 23,900 \Omega_m h^2$.

\[
D_H(z_{eq}) = R_0 r_H(z_{eq}) \approx 16(\Omega_m h^2)^{-1} \text{Mpc}
\]

Predicts hierarchical sequence of structure formation (smallest first).
##### Dark Matter and the Power Spectrum

- **Transfer Functions:**
  - Can quantify all this with the transfer function, $T(k)$:

$$\Delta_k^2 (z = 0) \propto T_k^2 \Delta_k^2 (z = \infty)$$
• No large oscillations or damping.
• Rules out a pure baryonic or pure HDM universe.
• Smooth power – expected for CDM-dominated universe.
• Detection of baryon oscillations – trace baryons.
Cosmological Parameters from 2dFGRS

Likelihood contours from the shape of the power spectrum:

Break scale:
Matter density:
\( \Omega_m h = 0.19 \pm 0.02 \)

Baryon oscillations:
Baryon fraction

\( = 0.18 \pm 0.06 \) (if \( n = 1 \))

So \( \Omega_m = 0.27 \ (h/0.7)^{-1} \)

\( \Omega_B = 0.04 \ (h/0.7)^{-1} \)
Observations: 2dFGRS Power-Spectrum

- Information about the amplitude of the power spectrum is confused, as we are looking at galaxies, not matter.

- We usually assume a linear relation between matter and density perturbations:

\[ \delta_{\text{galaxies}} = b \delta_{\text{matter}} \]

So amplitude of galaxy clustering mixes primordial power and process of galaxy formation.

b-bias parameter
Structure Formation in a CDM Universe
Cosmological Inflation

- Standard Model of Cosmology explains a lot (expansion, BBN, CMB, evolution of structure) but does not explain:
  - Origin of the Expansion:
    Why is the Universe expanding at $t=0$?
  - Flatness Problem:
    Why is $\Omega \sim 1$?
Cosmological Inflation

- Standard Model of Cosmology explains a lot (expansion, BBN, CMB, evolution of structure) but does not explain:
  - Horizon Problem: Why is the CMB so uniform over large angles, when the causal horizon is 1 degree?
  - Structure Problem: What is the origin of the structure?
Cosmological Inflation

• Lets tackle the horizon problem first.
  • Recall for $R \sim t^{1/2}$ we have a particle horizon:

• But if $R \sim t^\alpha$, $\alpha > 1$, can causally connect universe:

• More generally $\ddot{R} > 0$

• This happens when:

\[
\ddot{R} = -\frac{4\pi G}{3} R \left( \rho + 3 \frac{p}{c^2} \right) > 0
\]

which we get from Vacuum Energy, $p_v = -\rho_v c^2$
In 1980 Alan Guth proposed that the Early Universe had undergone acceleration, driven by vacuum energy.
He called this Cosmological Inflation.

- Inflates a small, uniform causal patch to the size of observable Universe.
- Explains why Universe (& CMB) looks so uniform.
Expansion Problem:

- Vacuum energy leads to acceleration of the Early Universe
- This powers the expansion (recall Eddington).

$$H^2 \propto \rho_v = \text{const}$$
$$\dot{R} = HR$$
$$R = R_0 e^{Ht}$$

- Need Inflation to end to start BB phase.
Cosmological Inflation

• The Flatness Problem: Recall that if we expand a model with curvature, it looks locally flat:

\[ H^2 = \frac{8\pi G \rho_V}{3} - \frac{c^2 k}{R^2} \]

\[ R \rightarrow \infty \]

\[ H^2 = \frac{8\pi G \rho_V}{3} \]

\[ k = 0 \]

• So Inflation predict \( \Omega = \Omega_m + \Omega_v = 1 \) to high accuracy.
• Compare with SN & galaxy clustering results.
• How much Inflation do we need?

• Usually assume Inflation happens at GUT era, $E_{\text{GUT}} \sim 10^{15}\text{GeV}$.

• So how large is the current horizon at the GUT era?

\[
d_H(\text{today}) = 6000h^{-1}\text{Mpc}
\]

\[
d_H(\text{GUT}) = d_H(\text{today}) /(1 + z_{\text{GUT}}),
\]

\[
1 + z_{\text{GUT}} = \frac{E_{\text{GUT}}}{E_{\text{CMB}}} = \frac{10^{15}\text{GeV}}{2.5 \times 10^{-13}\text{GeV}} \approx 10^{27}
\]

\[
d_H(\text{GUT}) \approx 10^{-24}\text{Mpc} \approx 10^{-2}\text{m}
\]

• But causal horizon at GUT era is just $d_{\text{GUT}} = ct_{\text{GUT}} = 3 \times 10^{-27}\text{m}$.

• So need to stretch GUT horizon by factor $a_{\text{Infl}} = 10^{29} \approx e^{60}$.
Lecture 16

Lecture Notes, PowerPoint notes, Tutorial Problems and Solutions are now available at: http://www.roe.ac.uk/~ant/Teaching/Astro%20Cosmo/index.html
Dynamics of Inflation

- We need a dynamical process to switch off inflation.
- Simplest models are based on scalar fields (spin-0), \( \phi \), the inflaton (e.g. \( \pi \)-mesons, Higgs bosons). No idea what \( \phi \) is…

- Must obey energy equation: \( E^2 = p^2c^2 + m^2c^4 \)
- Quantize to get Klein-Gordon equation: \( p = -i\hbar \nabla, \quad E = i\hbar \partial_t \)

\[
\ddot{\phi} - c^2 \nabla^2 \phi = -(m^2c^4\hbar^{-2})\phi
\]

- Assume \( \phi \) is uniform and add expansion term, 3H:

\[
\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \quad V' = \frac{dV}{d\phi}
\]

- Equation of a particle in a potential \( V=(m^2c^4\hbar^{-2})\phi^2/2 \).
Dynamics of Inflation

Evolution of a scalar field, $\phi$, in a potential $V(\phi)$.

Energy of scalar field:

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)$$

Assume Slow Roll:

$$\dot{\phi}^2 \ll V$$
$$\rho \approx V \approx \text{const.}$$

So like a vacuum energy with pressure

$$p = -\rho c^2$$

Drives acceleration of expansion.
Dynamics of Inflation

- The End of Inflation.
- Eventually the inflaton will reach the bottom of the potential and oscillate. \( V(\phi) \)
- No longer slow-rolling.
- \( \dot{\phi}^2 \gg V \)
- The Inflaton can then decay into other particles and radiation, re-heating Universe for radiation-domination.
Dynamics of Inflation

The Origin of Structure in Inflation.

The evolution of the inflaton has quantum fluctuations

\[ \varphi = \varphi_{\text{classical}} + \delta \varphi_{\text{quantum}} \]

Fluctuations due to Hawking Radiation (c.f. Black Holes).

\[ \delta \varphi = \frac{\hbar H}{2\pi} \]

So the universe expands at different rates, leading to density perturbations:

\[ \delta = \frac{\delta \rho}{\rho} = -3 \frac{\delta R}{R} = -3H \delta t = -3H \frac{\delta \varphi}{\dot{\varphi}} \]

\[ \delta t = \frac{\delta \varphi}{\dot{\varphi}} \]
The Origin of Structure in Inflation.

During Slow-Roll \(|\dot{\phi}| < |V'|\) equation of motion simplifies:

\[ 3H \dot{\phi} = -V', \quad \Rightarrow \quad \dot{\phi} = -\frac{V'}{3H} \]

The induced density field is:

\[ \delta = -3H \frac{\delta \phi}{\dot{\phi}} = \frac{9 \hbar H^3}{2\pi V'} \propto \frac{V^{3/2}}{V'} \]

The density power spectrum is

\[ P(k) \propto \langle \delta^2 \rangle \propto \frac{V^3}{V'^2} \]

Exponential expansion generates a fractal in the potential field: so spectral index is \( n = 1 \).
A Gravitational Wave Background from Inflation

- Structure created by freezing in quantum fluctuations during an inflationary epoch. Leaves imprint in structure.

Gravitational wave spectrum:

$$P_{GW}(k) \sim \delta \varphi^2 V$$

Quantum fluctuations of massless virtual particles (inflaton, gravitational waves) excited by expansion.

Frozen-in structure & gravitational waves
Inflation can happen lots of times, producing a Multiverse.

So somewhere life will appear.
The Cosmic Microwave Background

Recombination

Plasma

T = 2.73K

Observer

z = 1000

z = infinity
The Cosmic Microwave Background

Recombination

Plasma

T = 2.73K

Observer

z = 1000

z = infinity
The Cosmic Microwave Background

- Recombination
- Causal horizon $\sim 1^\circ$
- Plasma
- Observer
- $T=2.73K$
- $z=1000$, $z = \text{infinity}$
• Recall in 1930’s George Gamow predicts an afterglow from the Big Bang.
• Formed at reionization at $z = 1100$.
• In 1960’s Sachs and Wolfe predicted there should be anisotropies due to potential perturbations.
Anisotropies first detected in 1992 by the COBE satellite.

• On large angular scales $\Delta T/T = 10^{-5}$

• Detail images from the Wilkinson Microwave Anisotropy Probe (WMAP) in 2003.

• First year data.

• Third year data expected any day now.
Spherical Harmonic Analysis of the CMB

- Expand fluctuations in CMB temperature field in spherical harmonics on the celestial sphere:

\[
\Delta T(\theta, \phi) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\theta, \phi)
\]
Spherical Harmonic Analysis of the CMB

• The squared harmonic modes of the temperature can be averaged to give a power spectrum:

\[
\langle |T_{\ell m}|^2 \rangle = C_{\ell}^{TT}
\]

where

\[
\frac{\ell (\ell + 1) C_{\ell}^{TT}}{2\pi} \approx \left( \frac{\Delta T}{T} \right)^2 (\theta \approx 2\pi / \ell)
\]
ΔT on super-horizon scales (>1°)

- Sachs-Wolfe Effect: Gravitational redshift due to photons climbing out of potential wells,

\[
\frac{\Delta T}{T} = \frac{1}{3} \Phi
\]

Φ Newtonian potential
ΔT on sub-horizon scales (<1°)

- Acoustic Oscillations: Baryons oscillate in and out of dark matter potential wells,

\[
\frac{\Delta T}{T} = \frac{4}{3} \frac{\delta \rho}{\rho} \approx A \cos(k \tau)
\]

\[\lambda \ll c t\]
$\Delta T$ on sub-horizon scales ($<1^\circ$)

- Photon Diffusion: Photons random walk out of potential wells,

$$\frac{\Delta T}{T} \propto \exp\left(-\ell^2 \theta_s^2 / 2\right)$$
The CMB Temperature Power Spectrum

- Sachs-Wolfe Effect
- Acoustic Peaks
- Diffusion Damping

- Universe is spatially flat ($\Omega_m + \Omega_v = 1$).
- Potential amplitude: $\Phi = 10^{-5}$
- Spectral slope $n \sim 1$
The peak of the CMB power spectrum is given by the horizon size at recombination. Hence it is a fixed “ruler”.

**Matter only universe:**

\[
D_H(z) = R_0 \int_z^\infty dr = \frac{2c}{H_0} \left[ \Omega_m (1 + z) \right]^{-1/2} \approx 18 \Omega_m h^2 \approx 22 Mpc
\]

\[
\theta_H \approx 1.8 \Omega_m^{-1/2} \text{ deg}
\]
Dark Matter and Vacuum Energy

- The CMB acoustic peak measure spatial curvature of Universe.
- Combine with supernova, galaxy clusters, galaxy clustering.
- Likelihood contours for $\Omega_m - \Omega_V$ plane.
  \[ \Omega_V = 0.7, \Omega_m = 0.3 \]
- So four independent methods converge on one model.
Putting it all together

The “Standard Model” of Cosmology:

- CMB acoustic peaks: \( \Omega_m + \Omega_v = 1 \)
- Supernova: \( 2\Omega_v - \Omega_m = 1.1 \)
- CMB + SN: \( \Omega_v = 0.7 \)
- Galaxy clusters and galaxy clustering: \( \Omega_m = 0.3 \)
- BBN and CMB: \( \Omega_B = 0.04 \)
- CMB Sachs-Wolfe effect: \( \Phi = 10^{-5} \)
- CMB + galaxy clustering: \( n \sim 1 \)
- HST key programme and CMB: \( h = 0.7 \)
- Age of Universe: \( 13.7 \pm 0.2 \text{ Gyrs} \)
Structure in the Universe with CDM

Amplitude of Structure, $\delta$
Open Questions and Speculations

• Need to explain this strange Universe.
  • What is vacuum energy (dark energy)?
  • Why is $\rho_V \sim (1 \text{ eV})^4 \sim \rho_m$?
    • New particle physics, change gravity?
  • What is the Cold Dark Matter?
    • CDM – neutralino in LHC?
  • Did inflation happen?
    • Detect gravity wave background?
  • How did galaxies form?
    • Watch them form at high-z?
A Multiverse?

- Inflation and superstring theory both predict a multiverse.
- Find $\sim 10^{120}$ universes...
- Life will appear only in those Universes with small vacuum energy.
Parallel Universe Cosmologies

- Speculative ideas like the Ekpyrotic Universe try to unify Dark Energy and Inflation.
The End