Fast iterative reconstruction methods in atmospheric tomography

Daniela Saxenhuber[1]

joint work with

Ronny Ramlau[1], Matthias Rosensteiner[2] , Andreas Obereder[3]

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[1]Industrial Mathematics Institute JKU Linz, Austria

[2]NRC Herzberg

[3]MathConsult GmbH, Austria

Overview

Overview

3-step approach

1. **Wavefront reconstruction** from WFS data s_{α_g} for each guide star direction α_g:

$$\Gamma \varphi_{\alpha_g} = s_{\alpha_g}$$

   e.g. use CuReD algorithm[4]

2. **Atmospheric tomography**: Reconstruct a discretized atmosphere $\mathbf{\Phi} = (\Phi^{(1)}, \ldots, \Phi^{(L)})$ from data φ_{α_g}

$$\mathbf{A}\mathbf{\Phi} = \varphi$$

   e.g. via Gradient-based method, Kaczmarz, CG, Backprojection, ...

3. **Shape of the deformable mirror**: projection of atmosphere $\mathbf{\Phi}$ onto DM $\mathbf{\Phi}_{DM}$

$$\mathbf{P}\mathbf{\Phi} = \mathbf{\Phi}_{DM}$$

[4] M. Rosensteiner, *Wavefront reconstruction for extremely large telescopes via CuRe with domain decomposition*. J. Opt. Soc. Am. A, Vol. 29, Nr. 11 (2012)

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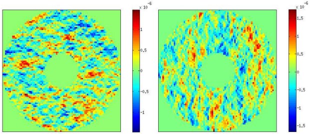
[4] M. Rosensteiner, *Wavefront reconstruction for extremely large telescopes via CuRe with domain decomposition*. J. Opt. Soc. Am. A, Vol. 29, Nr. 11 (2012)

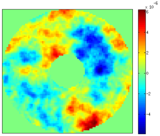
3-step approach

WFS measurements s^x and s^y



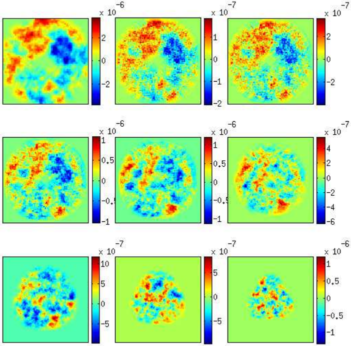
$\downarrow$ *Wavefront Reconst.*

incoming wavefront



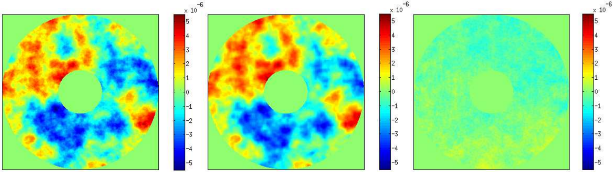
$\xrightarrow{\textit{Atm.Tom.}}$

turbulent layers



$\swarrow$ *Projection step*

incoming screen   DM shape   residual



Advantages of the 3-step approach

- very fast algorithm
- uses no matrix-vector-multiplications
- usable for (linearised) SH WFS and Pyramid WFS [5]
- different algorithms for atmospheric tomography developed
- efficient treatment of the LGS deficiencies tip/tilt indetermination and cone effect in the atmospheric tomography step
- efficient projection step, in particular for MCAO with e.g. 5x5 optimization directions
- all steps are highly parallelizable, CuReD algorithm even pipelineable
- low overall computational complexity: $O(n)$
- very good quality results for NGS systems and also for small FoV systems with spot elongation

[5] I. Shatokhina et al., *Preprocessed cumulative reconstructor with domain decomposition: a fast wavefront reconstruction method for pyramid wavefront sensor.* Appl. Opt., Vol. 52, Nr.12 (2013)

Wavefront reconstruction

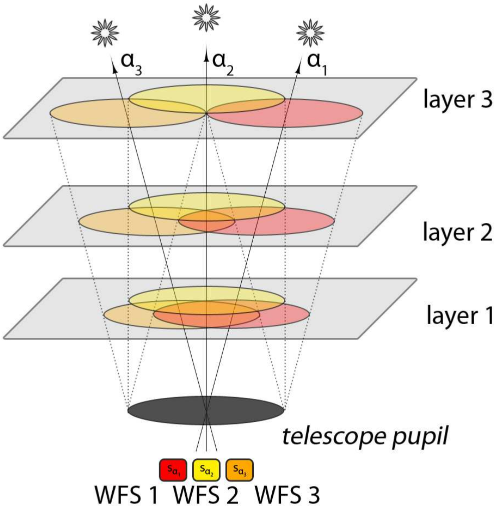
- Solve $\Gamma\varphi_{\alpha_g} = s_{\alpha_g}$, with $[s^x_{\alpha_g} s^y_{\alpha_g}]$
- Shack-Hartmann Wavefront Sensor (SH-WFS) for each guide star direction $\alpha_g,\ g = 1, \ldots, G + N$
- SH-operator:

$$\Gamma : H^1(\Omega_D) \to \ \mathbb{R}^{2\#sub}$$
$$s^x_{\alpha_g} = \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \frac{\partial \varphi_{\alpha_g}}{\partial x}\, d(x, y)\,,$$
$$s^y_{\alpha_g} = \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \frac{\partial \varphi_{\alpha_g}}{\partial y}\, d(x, y)\,,$$

$\varphi_{\alpha_g} \ldots$ incoming wave-front
$\#sub \ldots$ number of active sub-apertures

Overview

Atmospheric tomography



Input:

- reconstructed incoming wavefronts φ_{α_g} on $\boldsymbol{\Omega}_D$ (aperture) from LGS $g = 1, \ldots, G$ and NGS $g = G+1, \ldots, G+N$

Goal:

- **fast** reconstruction of turbulence layers $\Phi^{(l)}$ on $\boldsymbol{\Omega}_l$, $l = 1, \ldots, L$

ill-posed inverse problem
$\Longrightarrow$ requires **regularization**.

Data model

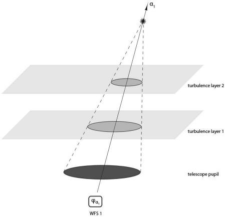
Light propagation through turbulence:

$$\mathbf{A}_{\boldsymbol{\alpha}_g} \quad : \quad \bigotimes_{l=1}^{L} L^2(\boldsymbol{\Omega}_l) \to L^2(\Omega_D), g = 1, \ldots, G+N$$

$$\mathbf{A}_{\boldsymbol{\alpha}_g}\boldsymbol{\Phi} \quad := \quad \sum_{l=1}^{L} \Phi^{(l)}(c_l\mathbf{r} + h_l\boldsymbol{\alpha}_g) = \varphi_{\alpha_g}(\mathbf{r})\,,\ \mathbf{r} \in \boldsymbol{\Omega}_D.$$

L^2-**adjoint** : $\gamma_l \ldots c_n^2$-profile at h_l

$$\mathbf{A}^*_{\boldsymbol{\alpha}_g}(\boldsymbol{\Psi}) \quad = \quad \left[\gamma_l \boldsymbol{\Psi}\left(\frac{1}{c_l}(\mathbf{r} - \alpha_g h_l)\right)\chi_{\Omega_D(\alpha_g h_l)}\left(\frac{1}{c_l}\mathbf{r}\right)))\right]^T_{l=1,\ldots,L}$$

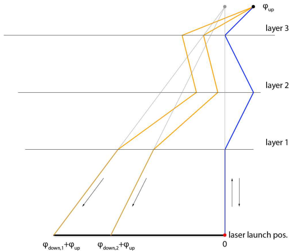


Cone Effect:

$$c_l := \begin{cases} 1\,, & \text{for NGS} \\ 1 - \frac{h_l}{h_{LGS}}, & \text{for LGS} \end{cases}$$

$h_{LGS} \ldots$ LGS height ~ 90km

Tip/tilt Indetermination



- remove wrong tip/tilt from incoming LGS wavefronts
- Tip/tilt removal operator

  $$\mathbf{\Pi} = (\underbrace{\Pi, \ldots, \Pi}_{G\times}, \underbrace{Id, \ldots, Id}_{N\times})$$

- $\Pi\varphi_{\alpha_g}(r) = \varphi_{\alpha_g}(r) - x \cdot t^x - y \cdot t^y$

Gradient tilt:

$$t^x = \frac{1}{|\Omega_D|}\int_{\Omega_D} \frac{\partial}{\partial x}\varphi_{\alpha_g}(x,y)d(x,y)$$
$$t^y = \frac{1}{|\Omega_D|}\int_{\Omega_D} \frac{\partial}{\partial y}\varphi_{\alpha_g}(x,y)d(x,y)$$

Zernike tilt:

$$t^x = \frac{1}{c|\Omega_D|}\int_{\Omega_D} x \cdot \varphi_{\alpha_g}(x,y)d(x,y)$$
$$t^y = \frac{1}{c|\Omega_D|}\int_{\Omega_D} y \cdot \varphi_{\alpha_g}(x,y)d(x,y)$$

Tip/Tilt reconstruction with tip/tilt sensor data[6]

- LGS measurements contain no tip/tilt information
- additional tip/tilt measurements $\mathbf{t}_{\beta_i}$ from natural guide stars

$$\mathbf{N}_{\beta_i}\mathbf{\Phi} \quad = \quad \mathbf{t}_{\beta_i} = \begin{pmatrix} t^x_{\beta_i} \\ t^y_{\beta_i} \end{pmatrix} \in \mathbb{R}^2 \ , i = 1, \cdots, N.$$

Gradient tilt:

$\mathbf{N}_{\beta_i} : \bigotimes_{l=1}^{L} H^1(\mathbf{\Omega}_l) \to \mathbb{R}^2$ and

$\mathbf{N}^*_{\beta_i} : \mathbb{R}^2 \to \bigotimes_{l=1}^{L} H^1(\mathbf{\Omega}_l)$

costly solution of PDE $\to$ use L^2-adjoint $\mathbf{A}^*_{\beta_i}$ as an approximation

Zernike tilt:

$\mathbf{N}_{\beta_i} : \bigotimes_{l=1}^{L} L^2(\mathbf{\Omega}_l) \to \mathbb{R}^2$, cheap adjoint:

$\mathbf{N}^*_{\beta_i} : \mathbb{R}^2 \to \bigotimes_{l=1}^{L} L^2(\mathbf{\Omega}_l)$ with

$$\mathbf{N}^*_{\beta_i}\mathbf{t}_{\beta_i}(\mathbf{r}) = \left(\begin{pmatrix} t^x_{\beta_i} \\ t^y_{\beta_i} \end{pmatrix} \cdot (\mathbf{r} - \beta_i h_l) \right)_{l=1}^{L}$$

[6] R. Ramlau, A. Obereder, R. Rosensteiner and D. Saxenhuber, *Efficient iterative tip/tilt reconstruction for atmospheric tomography*. Inverse Probl. Sci. En. (2014)

Kaczmarz method[7]

Solve $\quad \mathbf{A}_{\boldsymbol{\alpha}_g}\mathbf{\Phi} = \varphi_{\alpha_g}\,, \quad$ for $\; g = 1, \ldots, G+N$

Algorithm 1 Kaczmarz iteration for LTAO and MOAO

Choose $\mathbf{\Phi}_0$
for $i = 1, \ldots$ **do**
  $\mathbf{\Phi}_{i,0} = \mathbf{\Phi}_{i-1}$
  for $n = 1, \ldots, N$ **do**
    $\mathbf{\Phi}_{i,n} = \mathbf{\Phi}_{i,n-1} + \tau_{i,n} \cdot \mathbf{A}^*_{\beta_n}\left(\varphi_{\beta_n} - \mathbf{A}_{\beta_n}\mathbf{\Phi}_{i,n-1}\right)$
  end for
  for $g = G+1, \ldots, G+N$ **do**
    $\mathbf{\Phi}_{i,g} = \mathbf{\Phi}_{i,g-1} + \tau_{i,g} \cdot \mathbf{A}^*_{\boldsymbol{\alpha}_g}\Pi\left(\varphi_{\alpha_g} - \mathbf{A}_{\boldsymbol{\alpha}_g}\mathbf{\Phi}_{i,g-1}\right)$
  end for
  $\mathbf{\Phi}_i = \mathbf{\Phi}_{i,G+N}$
end for

Computational complexity: $\sim ((G+N)\cdot(18\cdot L - 14) + 2\cdot G)\cdot n$

[7] R. Ramlau and M. Rosensteiner, *An efficient solution to the atmospheric turbulence tomography problem using Kaczmarz iteration*. Inverse Problems, Vol.28, Nr. 9 (2012)

Kaczmarz method[7] with tip/tilt sensor data

Solve $\mathbf{A}_{\boldsymbol{\alpha}_g}\boldsymbol{\Phi} = \varphi_{\alpha_g}\,,$ for $g = 1, \ldots, G$

$\mathbf{N}_{\beta_n}\boldsymbol{\Phi} = \mathbf{t}_{\beta_n}\,,$ for $n = 1, \ldots, N$

Algorithm 2 Kaczmarz iteration for LTAO and MOAO

Choose $\boldsymbol{\Phi}_0$
for $i = 1, \ldots$ **do**
  $\boldsymbol{\Phi}_{i,0} = \boldsymbol{\Phi}_{i-1}$
  for $n = 1, \ldots, N$ **do**
    $\boldsymbol{\Phi}_{i,n} = \boldsymbol{\Phi}_{i,n-1} + \tau_{i,n} \cdot \mathbf{N}^*_{\beta_n}\left(\mathbf{t}_{\beta_n} - \mathbf{N}_{\beta_n}\boldsymbol{\Phi}_{i,G+n-1}\right)$
  end for
  for $g = G+1, \ldots, G+N$ **do**
    $\boldsymbol{\Phi}_{i,g} = \boldsymbol{\Phi}_{i,g-1} + \tau_{i,g} \cdot \mathbf{A}^*_{\boldsymbol{\alpha}_g}\Pi\left(\varphi_{\alpha_g} - \mathbf{A}_{\boldsymbol{\alpha}_g}\boldsymbol{\Phi}_{i,g-1}\right)$
  end for
  $\boldsymbol{\Phi}_i = \boldsymbol{\Phi}_{i,G+N}$
end for

$\rightarrow$ same strategy for other methods with tip/tilt sensor data

[7] R. Ramlau and M. Rosensteiner, *An efficient solution to the atmospheric turbulence tomography problem using Kaczmarz iteration*. Inverse Problems, Vol.28, Nr. 9 (2012)

Gradient-based method for LTAO and MOAO

$$\text{Solve } \mathbf{A}\mathbf{\Phi} = \begin{pmatrix} \mathbf{A}_{\boldsymbol{\alpha}_1} \\ \vdots \\ \mathbf{A}_{\boldsymbol{\alpha}_{G+N}} \end{pmatrix} \mathbf{\Phi} = \begin{pmatrix} \varphi_{\alpha_1} \\ \vdots \\ \varphi_{\alpha_{G+N}} \end{pmatrix} = \varphi\,.$$

Approach without noise models:

Least-squares functional:

$$J(\mathbf{\Phi}) = \|\mathbf{A}\mathbf{\Phi} - \varphi\|^2_{[L^2(\mathbf{\Omega}_D)]^{G+N}} \to \min\ .$$

Approach with noise models:

Tikohonov functional:

$$J(\mathbf{\Phi}) = \|\mathbf{A}\mathbf{\Phi} - \varphi\|^2_{\overline{C_\eta}^{-1}} + \alpha_\Phi \|\mathbf{\Phi}\|^2_{C_\Phi^{-1}} \to \min\ .$$

C_Φ ... statistics of atmosphere, $\overline{C_\eta}$ ... noise on phases due to spot elongation

Noise modelling

- Statistics of the atmosphere: turbulence model implemented using DWT (→ see talk M.Yudytskiy)

$$C_\Phi = \begin{pmatrix} C_{\Phi^{(1)}} & & 0 \\ & \ddots & \\ 0 & & C_{\Phi^{(L)}} \end{pmatrix}$$

- Noise statistics due to spot elongation:
  - only for LGS
  - noisy SH sensor data $s^\delta_{\alpha_g} = s_{\alpha_g} + C^{1/2}_{\alpha_g}\eta$, with η white noise.
  - Noise model for incoming wavefronts:

$$\begin{aligned} \varphi^\delta_{\alpha_g} &= \Gamma^\dagger(s_{\alpha_g} + C^{1/2}_{\alpha_g}\eta), \text{ with } \Gamma \text{ the SH-operator.} \\ \text{cov}(\varphi_{\alpha_g}) &= (\Gamma^\dagger C^{1/2}_{\alpha_g})(\Gamma^\dagger C^{1/2}_{\alpha_g})^T = \Gamma^\dagger C_{\alpha_g}(\Gamma^\dagger)^T =: \overline{C_{\alpha_g}} \\ \overline{C_\eta} &:= \text{diag}(\overline{C_{\alpha_1}}, \dots, \overline{C_{\alpha_G}}, \underbrace{\sigma^2 I, \dots, \sigma^2 I}_{N \text{ times}}) \end{aligned}$$

  - noise variance of GS σ^2 can also be chosen differently for each GS

Approach with noise models

$$J(\mathbf{\Phi}) = \|\mathbf{A}\mathbf{\Phi}-\varphi\|^2_{\overline{C_\eta}^{-1}} + \alpha_\Phi \|\mathbf{\Phi}\|^2_{C_\Phi^{-1}} \to \min ,$$

$$J'(\mathbf{\Phi}) = -2\mathbf{A}^*\mathbf{\Pi}\overline{C_\eta}^{-1}\mathbf{\Pi}(\varphi - \mathbf{A}\mathbf{\Phi}) + 2\alpha_\Phi C_\phi^{-1}\mathbf{\Phi} =: -\mathbf{d}$$

$$= -2\left(\sum_{g=1}^{G}\mathbf{A}^*_{\boldsymbol{\alpha}_g}\Pi\overline{C_{\alpha_g}}^{-1}\Pi(\varphi_{\alpha_g}-\mathbf{A}_{\boldsymbol{\alpha}_g}\mathbf{\Phi}) + \sum_{g=G+1}^{G+N}\mathbf{A}^*_{\boldsymbol{\alpha}_g}(\varphi_{\alpha_g}-\mathbf{A}_{\boldsymbol{\alpha}_g}\mathbf{\Phi}) - \alpha_\Phi C_\phi^{-1}\mathbf{\Phi}\right)$$

Gradient–based iteration with (heuristic) stepsize τ:

$$\mathbf{\Phi}_{j+1} = \mathbf{\Phi}_j + \tau_j \cdot \gamma. * \mathbf{d}_j$$

$$\tau_j = \min_{t\in[0,\infty)} J(\mathbf{\Phi}_j + t\mathbf{d}_j)$$

$$= \frac{\frac{1}{2}\langle \mathbf{d}_j, \mathbf{d}_j\rangle}{\langle \mathbf{\Pi}\overline{C_\eta}^{-1}\mathbf{\Pi}\mathbf{A}\mathbf{d}_j, \mathbf{A}\mathbf{d}_j\rangle_{[L^2(\Omega_D)]^{G+N}} + \alpha_\Phi\langle C_\Phi^{-1}\mathbf{d}_j, \mathbf{d}_j\rangle}$$

Algorithm 3 Gradient–based method for LTAO and MOAO

Choose $\mathbf{\Phi}_0$.
for $i = 1, \dots$ **do**
  $\mathbf{\Phi}_i = \mathbf{\Phi}_{i-1}$,
  for $g = 1, \dots, G$ **do**
    residual$_g$ = $[\ \overline{C_{\alpha_g}}^{-1}\Pi\ ]\ (\varphi_{\alpha_g} - \mathbf{A}_{\boldsymbol{\alpha}_g}\mathbf{\Phi}_i)$ *[if noise models included]*
    gradient$_g$ = $(\mathbf{A}_{\boldsymbol{\alpha}_g})^*\Pi$ residual$_g$
    $\mathrm{r}_i = \mathrm{r}_i + L\cdot$ residual$_g$
    $\mathrm{g}_i = \mathrm{g}_i +$ gradient$_g$
  end for
  for $g = G+1, \dots, G+N$ **do**
    residual$_g$ = $\varphi_{\alpha_g} - \mathbf{A}_{\boldsymbol{\alpha}_g}\mathbf{\Phi}_i$
    gradient$_g$ = $(\mathbf{A}_{\boldsymbol{\alpha}_g})^*$ residual$_g$
    $\mathrm{r}_i = \mathrm{r}_i + L\cdot$ residual$_g$
    $\mathrm{g}_i = \mathrm{g}_i +$ gradient$_g$
  end for
  $[\ \mathrm{g}_i = \mathrm{g}_i - \alpha_\Phi \cdot C_\Phi^{-1}\cdot \mathrm{g}_i\ ]$ *[if noise models included]*
  stepsize = $(\mathrm{g}_i^T \cdot \mathrm{g}_i)/(\mathrm{r}_i^T \cdot \mathrm{r}_i\ [\ + \alpha_\Phi \cdot g_i^T \cdot C_\Phi^{-1} g_i\ ]\ )$ *[if noise models included]*
  $\mathbf{\Phi}_i = \mathbf{\Phi}_i +$ stepsize $\cdot\gamma\cdot \mathrm{g}_i$
end for

Computational Complexity: $\sim ((G+N)\cdot(17\cdot L - 13) + 2\cdot G + (3\cdot L + 2))\cdot n$

Overview

MOAO quality results on OCTOPUS

Test case setting:

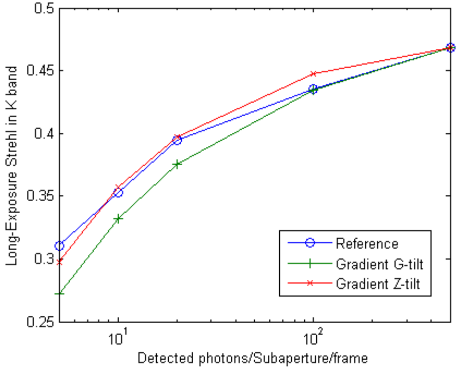
- telescope diameter: 42m
- 9 Shack-Hartmann WFSs (84×84)
- 6 LGS (diam 7.5 arcmin), 3 NGS (diam 10 arcmin)
- 1 ground DM, direction of interest: zenith
- ESO Standard atmosphere, $r_0 = 12.9$cm

Reconstruction:

- 9 reconstruction layers
- input gain (for LGS and NGS separately) / output gain possible
- open loop control

MOAO reference case results

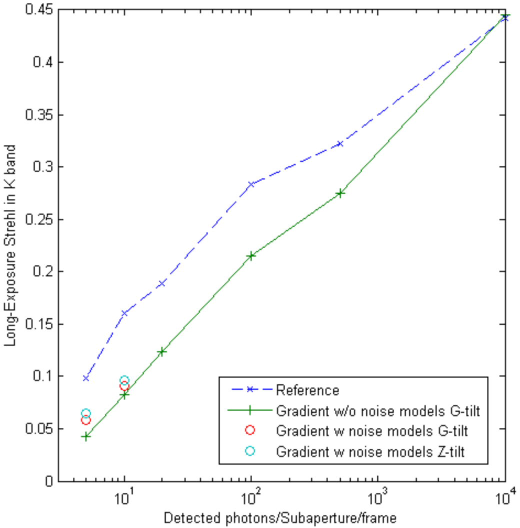
- tip/tilt indetermination and cone effect considered
- no noise due to spot elongation assumed
- Gradient method on least squares functional
- no noise models needed
- reference results: FRIM (by ESO)



	G-tilt	gain	Z-tilt	gain
500ph	30it	1.0	20it	1.0
100ph	30it	0.7	15it	0.8
20ph	30it	0.4	15it	0.4
10ph	20it	0.3	15it	0.3
5ph	20it	0.2	15it	0.3

MOAO with spot elongation

- Gradient method without noise models:
    - good in highflux
    - no comparable quality in lowflux regime reached
- Gradient method with noise models: ongoing work for lowflux regime
- small improvements using Z-tilt



Overview

Exploit flexibility of 3-step approach

Idea: Combine

Wavelet wavefront reconstructor ($\rightarrow$ talk M. Yudytsdkiy)

and

Gradient-based method for atmospheric tomography

- MOAO ELONG 100ph (reference: 28.3)

  CuReD + Gradient method w/o noise models: 21.53

  $\downarrow$

  Wavelet wf reconstr. + Gradient method w/o noise models: 24.37

- MOAO ELONG 5ph (reference 9.88)

  CuReD + Gradient method w/o noise models: 5.8

  $\downarrow$

  Wavelet wf reconstr. + Gradient method w/o noise models: 7.52

Summary

- 3-step approach instead of MVM
- Kaczmarz and Gradient-based method for atmospheric tomography
- including tip/tilt indetermination, cone effect, spot elongation
- fast, well parallelizable, flexibly combinable methods
- Quality results on OCTOPUS for MOAO system

Thank you for your attention!