FOURIER ANALYSIS: LECTURE 17

10 Partial Differential Equations and Fourier methods

The final element of this course is a look at partial differential equations from a Fourier point of view. For those students taking the 20-point course, this will involve a small amount of overlap with the lectures on PDEs and special functions.

10.1 Examples of important PDEs

PDEs are very common and important in physics. Here, we will illustrate the methods under study with three key examples:

$$\text{The wave equation}: \quad \nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \tag{10.205}$$

$$\text{The diffusion equation}: \quad \nabla^2\psi = \frac{1}{D}\frac{\partial\psi}{\partial t} \tag{10.206}$$

$$\text{Schrödinger's equation}: \quad \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{10.207}$$

These are all examples in 3D; for simplicity, we will often consider the 1D analogue, in which $\psi(\boldsymbol{r}, t)$ depends only on x and t, so that ∇^2 is replaced by $\partial^2/\partial x^2$.

10.1.1 The wave equation

A simple way to see the form of the wave equation is to consider a single plane wave, represented by $\psi = \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$. We have $\nabla^2\psi = -k^2\psi$, and $(\partial^2/\partial t^2)\psi = -\omega^2\psi$. Since $\omega/|\boldsymbol{k}| = c$, this one mode satisfies the wave equation. But a general ψ can be created by superposition of different waves (as in Fourier analysis), so ψ also satisfies the equation. Exactly the same reasoning is used in deriving Schrödinger's equation. Here we use de Broglie's relations for momentum and energy:

$$\boldsymbol{p} = \hbar\boldsymbol{k}; \quad E = \hbar\omega. \tag{10.208}$$

Then the nonrelativistic energy-momentum relation, $E = p^2/2m + V$ becomes $\hbar\omega = (\hbar^2/2m)k^2 + V$. A single wave therefore obeys Schrödinger's equation, and by superposition and completeness, so does a general ψ.

10.1.2 The diffusion equation

The diffusion equation is important because it describes how heat and particles get transported, (typically) under conduction. For completeness, here is a derivation – although this is non-examinable in this course.

The heat flux density (energy per second crossing unit area of surface) is assumed to be proportional to the gradient of the temperature (which we will call u, as T is conventionally used for the separated function $T(t)$):

$$f(x,t) = -\lambda \frac{\partial u(x,t)}{\partial x}, \tag{10.209}$$

where λ is a constant (the thermal conductivity). The minus sign is there because if the temperature gradient is positive, then heat flows towards negative x.

Now consider a thin slice of width δx from x to $x + \delta x$: there is energy flowing in at the left which is (per unit area) $f(x)$, and energy is flowing out to the right at a rate $f(x + \delta x)$. So in time δt, for unit area of surface, the energy content increases by an amount

$$\delta Q = \delta t \; [f(x) - f(x + \delta x)] \simeq -\delta t \, \delta x \, (\partial f / \partial x), \tag{10.210}$$

where we have made a Taylor expansion $f(x + \delta x) = f(x) + \delta x \, (\partial f / \partial x) + O([\delta x]^2)$. This heats the gas. The temperature rise is proportional to the energy added per unit volume,

$$\delta u = c \frac{\delta Q}{\delta V} \tag{10.211}$$

where the constant of proportionality c here is called the specific heat capacity. $\delta V = \delta x$ is the volume of the slice (remember it has unit area cross-section). Dividing by Δt then gives the diffusion equation:

$$\delta u = -c \left[\frac{\partial f}{\partial x} \delta x \right] \delta t \frac{1}{\delta V} = c\lambda \frac{\partial^2 u}{\partial x^2} \delta t \quad \Rightarrow \quad \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{10.212}$$

(for a constant $\kappa = c\lambda$). We often write $\kappa = 1/D$, where D is called the *diffusion coefficient*.

In 3D, this generalises to

$$\frac{\partial u}{\partial t} = \kappa \, \nabla^2 u. \tag{10.213}$$

The heat transport relation $f = -\lambda(\partial u/\partial x)$ takes a vector form $\boldsymbol{f} = -\lambda \boldsymbol{\nabla} u$, which is just a flow in the direction of maximum temperature gradient, but otherwise identical to the 1D case. When there is a flux-density vector in 3D, the corresponding density, ρ, obeys the *continuity equation*, $\boldsymbol{\nabla} \cdot \boldsymbol{f} = -\partial \rho / \partial t$. Since the change in temperature is c times the change in heat density, this gives the above 3D heat equation.

10.2   Solving PDEs with Fourier methods

The Fourier transform is one example of an integral transform: a general technique for solving differential equations.

Transformation of a PDE (e.g. from x to k) often leads to simpler equations (algebraic or ODE typically) for the integral transform of the unknown function. This is because spatial derivatives turn into factors of ik. Similar behaviour is seen in higher numbers of dimensions. When ψ is a single Fourier mode

$$\text{1D}: \quad \frac{\partial}{\partial x}\psi \to ik\psi; \quad \frac{\partial^2}{\partial x^2}\psi \to -k^2\psi \tag{10.214}$$

$$\text{3D}: \quad \boldsymbol{\nabla}\psi \to i\boldsymbol{k}\psi; \quad \nabla^2\psi \to -k^2\psi. \tag{10.215}$$

These simpler equations are then solved and the answer transformed back to give the required solution. This is just the method we used to solve ordinary differential equations, but with the

difference that there is still a differential equation to solve in the untransformed variable. Note that we can choose whether to Fourier transform from x to k, resulting in equations that are still functions of t, or we can transform from t to ω, or we can transform both. Both routes should work, but normally we would choose to transform away the higher derivative (e.g. the spatial derivative, for the diffusion equation).

The FT method works best for infinite systems. In subsequent lectures, we will see how Fourier series are better able to incorporate boundary conditions.

10.2.1 Example: the diffusion equation

As an example, we'll solve the diffusion equation for an *infinite system*.

$$\frac{\partial^2 n(x,t)}{\partial x^2} = \frac{1}{D}\frac{\partial n(x,t)}{\partial t}. \tag{10.216}$$

The diffusion coefficient D is assumed to be independent of position. This is important, otherwise the FT method is not so useful. The procedure is as follows:

- FT each side:
  - Multiply both sides by e^{-ikx}
  - Integrate over the full range $-\infty < x < \infty$.
  - Write the (spatial) FT of $n(x,t)$ as $\tilde{n}(k,t)$

- Pull the temporal derivative outside the integral over x

- Use Eqn. (3.33) with $p = 2$ to get:

$$(ik)^2 \tilde{n}(k,t) = \frac{1}{D}\frac{\partial \tilde{n}(k,t)}{\partial t} \tag{10.217}$$

- This is true for each value of k (k is a continuous variable). This is a partial differential equation, but let us for now fix k, so we have a simple ODE involving a time derivative, and we note that $d(\ln \tilde{n}) = d\tilde{n}/\tilde{n}$, so we need to solve

$$\frac{d\ln\tilde{n}}{dt} = -k^2 D. \tag{10.218}$$

  Its solution is $\ln\tilde{n}(k,t) = -k^2 Dt + \text{constant}$. Note that the constant can be different for different values of k, so the general solution is

$$\tilde{n}(k,t) = \tilde{n}_0(k)\, e^{-Dk^2 t}. \tag{10.219}$$

  where $\tilde{n}_0(k) \equiv \tilde{n}(k, t = 0)$, to be determined by the initial conditions.

- The answer (i.e. general solution) comes via an inverse FT:

$$n(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{n}(k,t)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{n}_0(k)\, e^{ikx - Dk^2 t}\ . \tag{10.220}$$
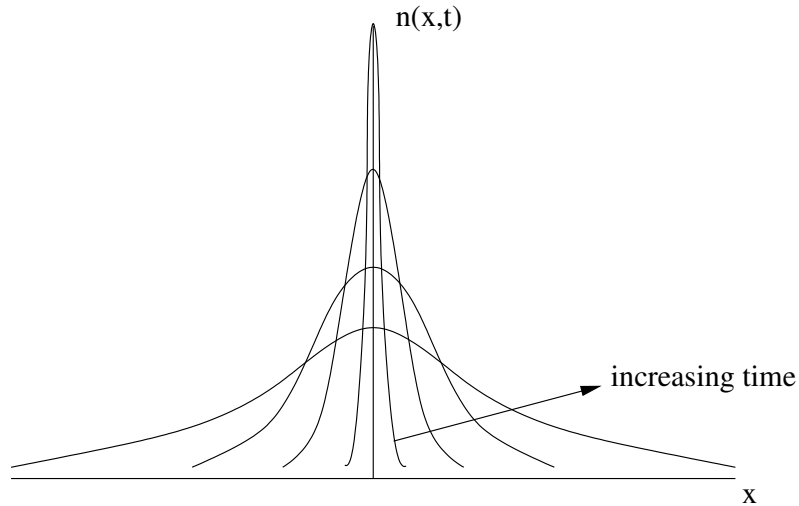
n(x,t)

increasing time

x

Figure 10.18: Variation of concentration with distance x at various diffusion times.

SPECIFIC EXAMPLE: We add a small drop of ink to a large tank of water (assumed 1-dimensional). We want to find the density of ink as a function of space and time, $n(x,t)$.

Initially, all the ink (S particles) is concentrated at one point (call it the origin):

$$n(x, t = 0) = S\,\delta(x) \tag{10.221}$$

implying (using the sifting property of the Dirac delta function),

$$\tilde{n}_0(k) \equiv \tilde{n}(k, 0) = \int_{-\infty}^{\infty} dx\, n(x, t = 0)\, e^{-ikx} = \int_{-\infty}^{\infty} dx\, \delta(x)\, e^{-ikx} = S. \tag{10.222}$$

Putting this into Eqn. (10.220) we get:

$$
\begin{aligned}
n(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{n}(k,t)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{n}_0(k)\, e^{ikx - Dk^2 t} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, S\, e^{ikx - Dk^2 t} = \frac{S}{\sqrt{2\pi}\sqrt{2Dt}} e^{-x^2/(4Dt)}\ .
\end{aligned} \tag{10.223}
$$

(we used the 'completing the square' trick that we previously used to FT the Gaussian). Compare this with the usual expression for a Gaussian,

$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{10.224}$$

and identify the width σ with $\sqrt{2Dt}$.

So, the ink spreads out with concentration described by a normalized Gaussian centred on the origin with width $\sigma = \sqrt{2Dt}$. The important features are:

- normalized: there are always S particles in total at every value of t

- centred on the origin: where we placed the initial drop

- width $\sigma = \sqrt{2Dt}$: gets broader as time increases

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- $\sigma \propto \sqrt{t}$: characteristic of random walk ('stochastic') process
- $\sigma \propto \sqrt{D}$: if we increase the diffusion constant D, the ink spreads out more quickly.

The solution $n(x,t)$ is sketched for various t in Fig. 10.18.

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10.3   Fourier solution of the wave equation

One is used to thinking of solutions to the wave equation being sinusoidal, but they don't have to be. We can use Fourier Transforms to show this rather elegantly, applying a partial FT ($x \to k$, but keeping t as is).

The wave equation is

$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \tag{10.225}$$

where c is the wave speed. We Fourier Transform w.r.t. x to get $\tilde{u}(k,t)$ (note the arguments), remembering that the FT of $\partial^2/\partial x^2$ is $-k^2$:

$$-c^2 k^2 \tilde{u}(k,t) = \frac{\partial^2 \tilde{u}(k,t)}{\partial t^2}. \tag{10.226}$$

This is a harmonic equation for $\tilde{u}(k,t)$, with solution

$$\tilde{u}(k,t) = A e^{-ikct} + B e^{ikct} \tag{10.227}$$

However, because the derivatives are partial derivatives, the 'constants' A and B can be functions of k. Let us write these arbitrary functions as $\tilde{f}(k)$ and $\tilde{g}(k)$, i.e.

$$\tilde{u}(k,t) = \tilde{f}(k) e^{-ikct} + \tilde{g}(k) e^{ikct}. \tag{10.228}$$

We now invert the transform, to give

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\tilde{f}(k) e^{-ikct} + \tilde{g}(k) e^{ikct} \right] e^{ikx} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ik(x-ct)} + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{g}(k) e^{ik(x+ct)} \\
&= f(x-ct) + g(x+ct)
\end{aligned}
$$

and f and g are arbitrary functions.

10.4   Fourier solution of the Schrödinger equation in 2D

Consider the time-dependent Schrödinger equation in 2D, for a particle trapped in a (zero) potential 2D square well with infinite potentials on walls at $x = 0, L$, $y = 0, L$:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\boldsymbol{x},t) = i\hbar \frac{\partial \psi(\boldsymbol{x},t)}{\partial t}. \tag{10.229}$$

For example, let us perform a FT with respect to $\boldsymbol{x}$. $\nabla^2 \to -\boldsymbol{k}.\boldsymbol{k} = -k^2 = -(k_x^2 + k_y^2)$, so

$$\frac{\hbar^2 k^2}{2m}\tilde{\psi}(\boldsymbol{k}, t) = i\hbar \frac{\partial \tilde{\psi}(\boldsymbol{k}, t)}{\partial t}. \tag{10.230}$$

We can integrate this with an integrating factor:

$$\tilde{\psi}(\boldsymbol{k}, t) = \tilde{\psi}(\boldsymbol{k}, 0)e^{-i\frac{\hbar k^2}{2m}t}. \tag{10.231}$$

The time-dependence of the wave is $e^{-i\omega t}$, or, in terms of energy $E = \hbar\omega$, $e^{-iEt/\hbar}$, where

$$E = \frac{\hbar^2 k^2}{2m}. \tag{10.232}$$

If the particle is trapped in the box, then the wavefunction must be zero on the boundaries, so the wavelength in the x direction must be $\lambda_x = 2L, L, 2L/3, \ldots$ i.e. $2L/m$ for $q = 1, 2, 3, \ldots$, or wavenumbers of $k_x = 2\pi/\lambda_x = q\pi/L$. Similarly, $k_y = r\pi/L$ for $r = 1, 2, \ldots$.

So the wavefunction is a superposition of modes of the form

$$\psi(\boldsymbol{x}, t) = A\sin\left(\frac{q\pi x}{L}\right)\sin\left(\frac{r\pi y}{L}\right)e^{-iEt/\hbar} \tag{10.233}$$

for integers q, r. A is a normalization constant to ensure that $\int\int |\psi^2|\, dx\, dy = 1$.

Each mode has an energy

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2(q^2 + r^2)}{2mL^2}. \tag{10.234}$$

For a square well, the energy levels are *degenerate* – different combinations of q and r give the same energy level.

11 Separation of Variables

We now contrast the approach of Fourier transforming the equations with another standard technique. If we have a partial differential equation for a function which depends on several variables, e.g. $u(x, y, z, t)$, then we can attempt to find a solution which is *separable* in the variables:

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t) \tag{11.235}$$

where X, Y, Z, T are some functions of their arguments, and we try to work out what these functions are. Examples of separable functions are xyz^2e^{-t}, $x^2\sin(y)(1 + z^2)t$, but not $(x^2 + y^2)zt$. Not all PDEs have separable solutions, but many physically important examples do.

Let us consider the one-dimensional wave equation (so we have only x and t as variables) as an example:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{11.236}$$

We try a solution $u(x, t) = X(x)T(t)$:

$$\frac{\partial^2(XT)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2(XT)}{\partial t^2}. \tag{11.237}$$

Now notice that on the left hand side, T is not a function of x, so can come outside the derivative, and also, since X is a function of x only, the *partial* derivative with respect to x is the same as the

ordinary derivative. A similar argument holds on the right hand side, where X is not a function of t, so

$$T\frac{d^2X}{dx^2} = \frac{X}{c^2}\frac{d^2T}{dt^2}. \tag{11.238}$$

The trick here is to divide by XT, to get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2}. \tag{11.239}$$

Now, the left hand side is not a function of t (only of x), whereas the right hand side is not a function of x (only of t). The only way these two independent quantities can be equal for all t and x is if they are both constant. The constant is called the *separation constant*, and let us call it $-k^2$. (Note that we aren't sure at this point that the separation constant is negative; if it turns out it is positive, we'll come back and call it k^2, or, alternatively, let k be imaginary). Hence the equation for X is (multiplying by X)

$$\frac{d^2X}{dx^2} = -k^2X. \tag{11.240}$$

You know the solution to this:

$$X(x) = A\exp(ikx) + B\exp(-ikx) \tag{11.241}$$

for constants A and B (alternatively, we can write X as a sum of sines and cosines).

The equation for T is

$$\frac{1}{c^2}\frac{d^2T}{dt^2} = -k^2T \tag{11.242}$$

which has solution

$$T(t) = C\exp(i\omega t) + D\exp(-i\omega t) \tag{11.243}$$

where $\omega = ck$. If we take in particular $B = C = 0$ and $A = D = 1$, we have a solution

$$u(x,t) = \exp[i(kx - \omega t)] \tag{11.244}$$

which we recognise as a sinusoidal wave travelling in the $+x$ direction. In general, we will get a mixture of this and $\exp[i(kx + \omega t)]$, which is a sinusoidal wave travelling in the negative x direction. We will also get the same exponentials with the opposite sign in the exponent. These could be combined into

$$u(x,t) = A\sin[(kx - \omega t) + \alpha] + B\sin[(kx - \omega t) + \beta], \tag{11.245}$$

which is a mixture of waves travelling in the two directions, with different phases.

IMPORTANT: Notice that we can add together any number of solutions with different values of the separation constant $-k^2$, and we will still satisfy the equation. This means that the full solution can be a more complicated non-periodic function, just as Fourier transforms allow us to express a general function as a superposition of periodic modes. In this case (as we saw above), the general solution of the 1D wave equation is

$$u(x,t) = f(x - ct) + g(x + ct), \tag{11.246}$$

for *any* (twice-differentiable) functions f and g.

In general to find which of the many solutions are possible in a given situation, we need to specify what the boundary conditions are for the problem, i.e. what restrictions are there on the solution at (for example) fixed values of x, or t. e.g. if the solution has to be periodic in x, we will have
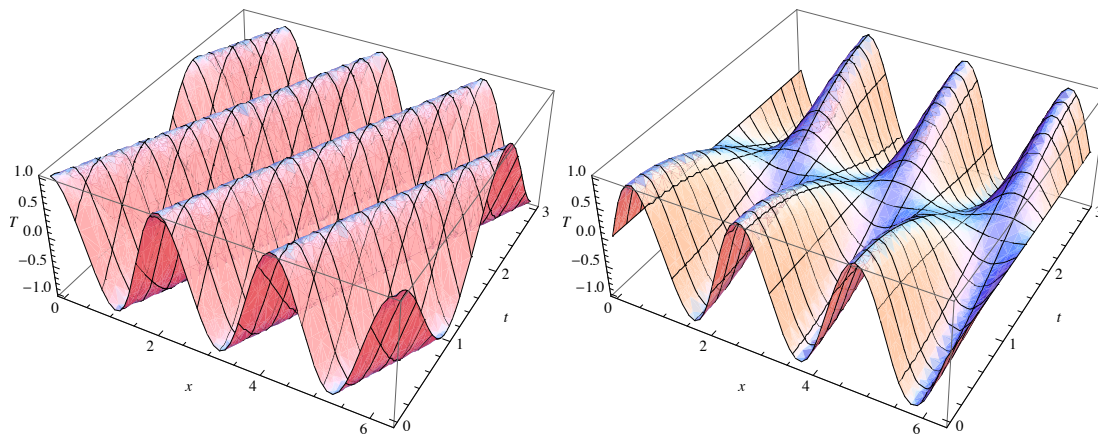
Figure 11.19: Contrasting the travelling-wave and standing-wave solutions to the wave equation.

exactly the same restrictions on k as in Fourier analysis. Note that the boundary conditions usually determine the solution.

Here is another example. If we require that u is zero at two boundaries $x = (0, \pi)$, and that at $t = 0$ the solution is a sin wave, $u(x, 0) = \sin(3x)$, then the solution is $u(x, t) = \sin(3x)\cos(3ct)$. This is a *standing wave*, which does not propagate, just varies its amplitude with time.

Note that we can write the standing wave solution as a superposition of waves travelling in opposite directions (with $ck = \omega$):

$$
\begin{aligned}
\sin(kx)\cos(\omega t) &= \frac{1}{2i}(e^{ikx} - e^{-ikx})\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \\
&= \frac{1}{4i}\left[e^{i(kx+\omega t)} + e^{i(kx-\omega t)} - e^{-i(kx-\omega t)} - e^{-i(kx+\omega t)}\right] \\
&= \frac{1}{2}\left[\sin(kx + \omega t) + \sin(kx - \omega t)\right]. \quad (11.247)
\end{aligned}
$$

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11.1 Solving the diffusion equation via separation of variables

Let us now try to solve the diffusion equation in 1D:

$$
\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}. \quad (11.248)
$$

We wish to find a solution with $u \to 0$ as $t \to \infty$. We try separating variables, $u(x, t) = X(x)T(t)$, to find

$$
\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{\kappa T}\frac{dT}{dt} = -\lambda^2 \quad (11.249)
$$

where we have written the separation constant as $-\lambda^2$. The equation for X is the same as we had before. This time, let us write the solution as sines and cosines:

$$
X(x) = A\sin(\lambda x) + B\cos(\lambda x). \quad (11.250)
$$

The equation for T is

$$\frac{dT}{dt} = -\lambda^2 \kappa T, \tag{11.251}$$

or

$$\frac{d \ln T}{dt} = -\lambda^2 \kappa \tag{11.252}$$

which has solution

$$T(t) = C \exp(-\lambda^2 \kappa t) \tag{11.253}$$

so we have a separable solution (absorbing the constant C into A and B)

$$u(x,t) = [A \sin(\lambda x) + B \cos(\lambda x)] \exp(-\lambda^2 \kappa t), \tag{11.254}$$

This tends to zero as $t \to \infty$ provided $\lambda^2 > 0$. This justifies our implicit assumption that $\lambda^2 > 0$ – otherwise we would have an unphysical diverging solution, which would not satisfy the boundary condition that $u \to 0$ as $t \to \infty$.

Note that we can add in solutions with different λ, and as usual the full solution will depend on the initial conditions. This begins to look like Fourier analysis. We certainly need to add at least a constant in order to make the solution physically sensible: as written, it allows for negative temperatures. But we can always add a constant to any solution of the diffusion equation; so when we speak of a boundary condition involving $u = 0$, this really means the temperature is at some uniform average value, which we do not need to specify.

11.2 Separation of variables with several dimensions

As a last example of solving equations via separation of variables, let us consider a more complicated situation, where we have 2 space dimensions, and time. Consider an infinite square column of side L which is initially (at $t = 0$) at zero temperature, $u(x, y, t = 0) = 0$. We ignore z as by symmetry there is no heat flowing along this direction. At $t = 0$ it is immersed in a heat bath at temperature T_0. We need to solve the heat equation

$$\nabla^2 u = \frac{1}{\kappa} \frac{\partial u}{\partial t}, \tag{11.255}$$

and we will look for separable solutions in the following form:

$$u(x, y, z, t) = T_0 + X(x)Y(y)T(t). \tag{11.256}$$

We do this because this will make it easier to absorb the boundary conditions. Although $u = 0$ throughout the block initially, it is $u = T_0$ at the *surface* of the block (at all times). Thus as written, $XYT = 0$ on the surface of the block at $t = 0$, which is a simpler boundary condition. We didn't have to choose this form, but it makes the working simpler. In any case, the differential equation is independent of T_0:

$$YT \frac{d^2 X}{d^2 x} + XT \frac{d^2 Y}{d^2 y} - \frac{XYZ}{\kappa} \frac{dT}{dt} = 0 \tag{11.257}$$

Dividing by XYT,

$$\frac{1}{X} \frac{d^2 X}{d^2 x} + \frac{1}{Y} \frac{d^2 Y}{d^2 y} - \frac{1}{\kappa T} \frac{dT}{dt} = 0 \tag{11.258}$$

Since the first term is not a function of y, t, and the second is not a function of x, t etc, we conclude that all the terms must be constant. e.g.

$$\frac{1}{\kappa T} \frac{dT}{dt} = -\lambda \quad \Rightarrow \quad T(t) \propto e^{-\lambda \kappa t}. \tag{11.259}$$

We next find the equation for X, by isolating terms which depend on x only:

$$\frac{1}{X}\frac{d^2 X}{d^2 x} = -\frac{1}{Y}\frac{d^2 Y}{d^2 y} - \lambda = -k_x^2 = \text{constant} \tag{11.260}$$

(the l.h.s. is not a function of y, z, the Y terms is not a function of x, hence they must be equal to another constant).

$$\frac{d^2 X}{d^2 x} = -k_x^2 X \quad \Rightarrow \quad X(x) = Ae^{ik_x x} + Be^{-ik_x x} \tag{11.261}$$

and similarly for Y – except that the equivalent wavenumber k_y must satisfy $k_x^2 + k_y^2 = \lambda$, from equation (11.260).

Now, as stated above, the terms we calculate here must be zero on the boundaries at $x = 0, L$, $y = 0, L$. Hence the solutions for X and Y must be sinusoidal, with the correct period, e.g.

$$X(x) \propto \sin\left(\frac{m\pi x}{L}\right) \tag{11.262}$$

for any integer m. Similarly for Y. So a separable solution is

$$u(x, y, t) = T_0 + C_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) e^{-\lambda_{mn}\kappa t} \tag{11.263}$$

where

$$\lambda_{mn} = \frac{\pi^2}{L^2}(m^2 + n^2). \tag{11.264}$$

Here we have identified the separation constants explicitly with the integers m, n, rewriting $\lambda = k_x^2 + k_y^2$.

Now we can add the separable solutions:

$$u(x, y, t) = T_0 + \sum_{m,n=0}^{\infty} C_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) e^{-\lambda_{mn}\kappa t}. \tag{11.265}$$

All that remains is to determine the constants C_{mn}. We use the initial condition that inside the volume $u = 0$ when $t = 0$ (when the exponential term is unity), so

$$\sum_{m,n=0}^{\infty} C_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) = -T_0. \tag{11.266}$$

This looks very much like a Fourier Series, and we can use the same trick of the orthogonality of the sin functions. Multiply by $\sin(m'\pi x/L)$ and integrate with respect to x, giving 0 unless $m = m'$, and $L/2$ if $m = m'$. Similarly for y, so

$$
\begin{aligned}
C_{mn}\left(\frac{L}{2}\right)^2 &= -T_0 \int_0^L \sin(m\pi x/L)dx \int_0^L \sin(n\pi y/L)dy \\
&= -T_0 \left[-\frac{L}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\right]_0^L \left[-\frac{L}{n\pi}\cos\left(\frac{n\pi y}{L}\right)\right]_0^L.
\end{aligned} \tag{11.267}
$$

The cosines are zero if m, n are even. If m, n are both odd, the right hand side is $4L^2/(mn\pi^2)$, from which we get

$$C_{mn} = \begin{cases} -16T_0/(\pi^2 mn) & m, n \text{ all odd} \\ 0 & \text{otherwise} \end{cases} \tag{11.268}$$

Finally the full solution is

$$u(x, y, t) = T_0 \left\{1 - \frac{16}{\pi^2}\sum_{m,n \text{ odd}} \frac{1}{mn}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\exp\left[-(m^2 + n^2)\frac{\kappa\pi^2}{L^2}t\right]\right\}. \tag{11.269}$$
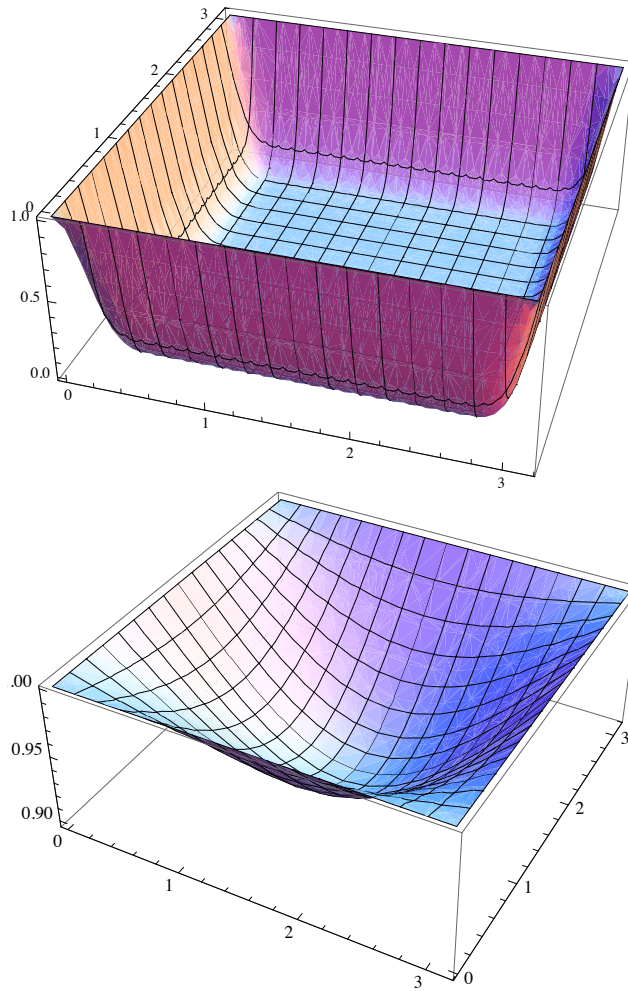
Figure 11.20: Temperature at an early time $t = 0.01$, for $T_0 = 1$, $\kappa = 1$ and $L = \pi$, and then at a late time $t = 1$.