MPhys Advanced Cosmology   2012/2013

Problem set 4

(1) Ionized material at low redshifts causes CMB photons to undergo Thomson scattering, which affects the observed CMB anisotropies.

(a) Show that the optical depth due to scattering between a redshift z and $z = 0$ is

$$\tau = \int \sigma_{\rm T}\, n_e\, d\ell_{\rm prop} = \int \sigma_{\rm T}\, n_e\, \frac{c}{H_0}\, \frac{dz}{(1+z)\sqrt{1-\Omega_m+\Omega_m(1+z)^3}}$$

(for a flat model).

(b) At low redshifts, structure formation causes reionization, so that $x = 1$ for $z < z_r$ Show that, for large z_r, the resulting optical depth is approximately

$$\tau = 0.04h\, \frac{\Omega_b}{\Omega_m} \left[\sqrt{1+\Omega_m z(3+3z+z^2)} - 1\right] \simeq 0.04h\, \frac{\Omega_b}{\Omega_m^{1/2}} z^{3/2}.$$

(c) Describe the effect of this scattering on the pattern of CMB anisotropies.

(2) The fractional density perturbation field, $\delta(\mathbf{x})$, is expressed as a Fourier integral;

$$\delta(\mathbf{x}) = \frac{V}{(2\pi)^3} \int \delta_k\ \exp(-i\mathbf{k}\cdot\mathbf{x})\ d^3k.$$

(a) Explain the meaning of the terms in this expression and justify its form, explaining the relation to the alternative form where the density field is written as a discrete Fourier series. How would the Fourier coefficients, δ_k be obtained if the density field was known?

(b) Explain how the expression in Fourier space makes it possible to find the gravitational potential that corresponds to a given density field. Discuss also the peculiar velocity field, which obeys the continuity equation $\nabla \cdot \mathbf{u} = -\partial\delta/\partial t$, and show how this may be solved if we assume $\mathbf{u}$ to be parallel to $\mathbf{k}$ in the growing mode. Explain the nature of a density disturbance with $\nabla \cdot \mathbf{u} = 0$, and give a physical reason why such modes to not grow with time.

(c) If the density field has a power spectrum $|\delta_k|^2 = Ak^n$, give an expression for the variance in density that results when the density field is filtered by convolution with a uniform sphere of radius R (dimensionless integrals need not be evaluated exactly). For what values of the index n will the result be finite? How do the required limits on n change if we require a finite variance in (a) gravitational potential; (b) peculiar velocity?

(3) Give a simple proof that surface brightness is conserved in Euclidean space. Now show that volume elements in phase space are Lorentz invariant, and hence that the relativistic expression of surface-brightness conservation is that I_ν/ν^3 is an invariant. Show from this that black-body radiation appears thermal to all observers. If this is so, how is it possible to use the microwave background to determine that the Earth has an absolute velocity of $\simeq 370\ \mathrm{km\,s^{-1}}$?

(4) Consider a uniformly expanding spherical clump of matter, of proper radius $\mathcal{R}(t)$ embedded in an expanding universe.

(a) Show that the expansion history can be expressed parametrically as

$$\mathcal{R} = A(1 - \cos\theta)$$
$$t = B(\theta - \sin\theta),$$

where $A^3 = GMB^2$.

(b) Expand this solution for small t, and show that the solution behaves as $\mathcal{R} \propto t^{2/3}$. If the clump is embedded in an Einstein-de Sitter universe, thus calculate the time dependence of the fractional deviation from the mean density inside the sphere, assuming that this is small.

(c) If we consider instead a proto-void, with $\delta < 0$, show that the parametric solution is still valid, provided trigonometric functions are replaced by their hyperbolic counterparts. Calculate the apparent Hubble parameter and density parameter for observers who live within the proto-void, and show that these are respectively larger and smaller than in the external Einstein-de Sitter universe. Could such an effect mimic cosmic acceleration?