Advanced Cosmology: Summer 2011

Section A Answer two Questions

A.1.

(a) Give a Newtonian justification for the Friedmann equation in the form $\dot{R}^2 - 8\pi G\rho R^2/3 = \text{const}$, defining carefully all terms that appear. What is the term on the rhs? **[5]**

(b) By considering conservation of energy, show that the the dependence of the cosmic density on scale factor is $d\ln\rho/d\ln a = -3(1+w)$, where $w \equiv P/\rho c^2$ and P is pressure. Hence derive an expression for $\ddot{R}$. If a 'deceleration parameter' is defined via $q \equiv -\ddot{R}R/\dot{R}^2$, show that $q = \Omega(1+3w)/2$, where Ω is the density parameter. **[6]**

(c) Show how the relation between comoving distance, D, and redshift, z, can be calculated if the expansion history is known in the form $H(z)$. If $D(z)$ is expanded in a Taylor series, explain why curvature only affects terms of order z^3 and higher. **[3]**

(d) Thus show that the angle subtended by an object of proper diameter L is $\theta = L/D(z)$, where $D(z) \simeq (c/H_0)(z - (3+q)z^2/2)$ and q is the deceleration parameter defined above (consider the Taylor series for $H(z)$). **[4]**

(e) Explain the meaning of the terms 'particle horizon' and 'event horizon'. Show that an event horizon exists in a flat universe containing matter and a cosmological constant, and that its value is a comoving distance of at most

$$(1-\Omega_m)^{-1/2}(c/H_0).$$

If we adopt $\Omega_m = 0.25$, the comoving distances to redshifts 1, 2, 3 are respectively 2385, 3795, 4690 h^{-1}Mpc; thus estimate the maximum present-day redshift of galaxies that can just be reached by light signals sent by us today. **[7]**

A.2.

(a) The equation of motion for a homogeneous scalar field evolving under the action of a potential $V(\phi)$ is

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0.$$

Write down without proof the expressions for the pressure and density of the scalar field, and hence give the equation for H, the hubble parameter during inflation, as a function of ϕ and its time derivative. Spatial curvature is normally neglected in inflationary models; explain why this is so. **[5]**

(b) Define the equation of state parameter w, and show that taking w close to -1 during inflation allows the equation of motion to be simplified to the slow-roll form. Show that the slow-roll equation of motion predicts

$$w = -1 + (2/3)\epsilon + O(\epsilon^2),$$

where the slow-roll parameter is $\epsilon \equiv m_{\rm P}^2 V'^2/(16\pi V^2)$. **[6]**

(c) Consider a power-law potential $V(\phi) \propto \phi^\alpha$. Assuming slow roll to apply, calculate the value of ϕ when inflation ends at $\epsilon = 1$, and when observed scales crossed the inflationary horizon, 60 e-folds before the end of inflation. Hence deduce the expected tensor-to-scalar ratio of perturbations to the microwave background, $r = 16\epsilon$. **[7]**

(d) When inflation terminates, the scalar field changes rapidly in comparison with the expansion timescale. If the potential is $V(\phi) = m^2\phi^2/2$, how that the scalar field then undergoes simple harmonic motion, and calculate the time average of the w parameter during this phase. Hence deduce the time dependence of the scale factor after the end of inflation, assuming that the scalar field continues to dominate the energy density. Explain briefly why this is unlikely to be true. **[7]**

A.3.

(a) Consider a uniformly expanding spherical clump of matter, of proper radius $\mathcal{R}(t)$ embedded in an expanding universe. Show that the expansion history can be expressed parametrically as

$$\begin{aligned} \mathcal{R} &= A(1 - \cos\theta) \\ t &= B(\theta - \sin\theta), \end{aligned}$$

where $A^3 = GMB^2$. **[5]**

(b) Expand this solution for small t, and show that the solution behaves as $\mathcal{R} \propto t^{2/3}$. If the clump is embedded in an Einstein-de Sitter universe, thus calculate the time dependence of the fractional deviation from the mean density inside the sphere, assuming that this is small. **[4]**

(c) Discuss the subsequent nonlinear evolution, and give expressions for the time at turnround and at collapse. What is the density contrast at turnround? If the universe is not Einstein-de Sitter, does this influence the evolution of $\mathcal{R}(t)$? **[3]**

(d) The equation describing the growth of density fluctuations in a matter-dominated expanding universe is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\bar{\rho}_m\,\delta,$$

where δ is the fractional density fluctuation, $\rho = \bar{\rho}_m(1+\delta)$, and $\bar{\rho}_m$ is the mean matter density. Show that the expression for $\delta(t)$ in an Einstein-de Sitter universe from part (b) satisfies this equation. **[5]**

(e) The above parts concern a proto-cluster region with $\delta > 0$. If we consider instead a proto-void, show that the parametric solution is still valid, provided trigonometric functions are replaced by their hyperbolic counterparts. Calculate the apparent Hubble parameter and density parameter for observers who live within the proto-void, and show that these are respectively larger and smaller than in the external Einstein-de Sitter universe. **[8]**