# Advanced Cosmology: Summer 2010

## Section A Answer two Questions

### A.1.

(a) Starting from the Friedmann equation, show that the Hubble parameter as a function of epoch can be written as

$$H^2(a) = H_0^2 \left( \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + \Omega_k a^{-2} \right),$$

defining carefully all terms that appear. Explain why the $\Omega_k$ term does not represent a contribution to the matter density. How does the equation chance if the vacuum energy is given a constant equation of state $w \neq -1$? Show that, provided $w < -1/3$, the universe will become vacuum dominated in the far future. **[5]**

(b) If $w < -1$, show from the Friedmann equation that the scale factor diverges at a finite time in the future. If the current matter density is neglected in comparison with the vacuum density, show that the time to this event is approximately

$$t - t_0 \simeq \frac{2}{3} H_0^{-1} |1 + w|^{-1} \Omega_v^{-1/2}.$$

**[6]**

(c) If the vacuum density is negative, prove that the expansion of the universe will always result in a maximum for the scale factor, followed by collapse to a big crunch, provided $w < -1/3$. For the case of a flat universe containing only matter and vacuum with $w = -1$, show that the Friedmann equation may be written as $\dot{a}^2 = -a^2 + 1/a$ with a suitable choice of time unit. Thus derive the exact expression for $a(t)$ and hence the time of maximum expansion and the time of the big crunch. The substitution $y = a^{3/2}$ should be useful. **[8]**

(d) Write down the integral for the relation between comoving distance and redshift. Discuss the use of this relation at the time of maximum expansion in the above recollapsing universe, and show that the leading dependence of redshift on distance is quadratic in distance. **[6]**

### A.2.

(a) Write down without proof the equation of motion for a scalar field evolving in an expanding universe under the action of a potential $V(\phi)$. Discuss why spatial derivatives of the field are normally neglected in models of inflationary cosmology. **[4]**

(b) For a homogeneous scalar field, what are the energy density and pressure? Thus give the condition under which inflationary expansion will proceed close to exponentially, and show how the equation of motion can be simplified under the slow-roll approximation. **[6]**

(c) Outline briefly how inflation terminates, distinguishing large-field and small-field inflation. Explain why, in the former case, the expectation is that the slow-roll parameter at the end of inflation is $\epsilon \simeq 1$, whereas small-field models allow $\epsilon \ll 1$ ($\epsilon \equiv m_{\rm P}^2 V'^2/16\pi V^2$). **[6]**

(d) One function of inflation is to stretch perturbations that cross the horizon scale during inflation ($c/H_{\rm inf}$) to a larger scale that can encompass the size (at reheating) of our current Hubble radius $c/H_0$. Assuming that inflation terminates suddenly, so that the relativistic energy density at reheating is equal to that during inflation, calculate the required stretch factor. Show that your result can be expressed approximately as $\epsilon^{1/4}\exp(62)$ by using the fluctuation amplitude $\delta_{\rm H} = H_{\rm inf}^2/2\pi|\dot{\phi}|$ and assuming that slow-roll applies before the end of inflation. You may assume that relativistic energy density scales exactly as $a^{-4}$, plus the following numerical values in natural units: $H_0 = 10^{-41.67}\, h$ GeV; present-day relativistic energy density $= 10^{-50.48}$ GeV$^4$. **[9]**

## A.3.

(a) The fractional density perturbation field can be expressed as a Fourier series, $\delta(\mathbf{x}) = \sum_{\mathbf{k}} \delta_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x})$, within a periodic normalization volume of side $L$. Show that the volume averaged variance in $\delta$ is $\langle \delta(\mathbf{x})^2 \rangle = \sum_{\mathbf{k}} |\delta_{\mathbf{k}}|^2$, and that the autocorrelation function of the field is $\langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle = \sum_{\mathbf{k}} |\delta_{\mathbf{k}}|^2 \exp(i\mathbf{k}\cdot\mathbf{r})$, where $\mathbf{r}$ is a constant spatial offset. Explain how Poisson's equation can be solved in Fourier space to give an expression for the gravitational potential, $\Phi$. **[6]**

(b) For density perturbations located in a screen at fixed distance from the observer, the gravitational deection of light by matter in the screen induces a bend angle given by the line-of-sight integral

$$\boldsymbol{\alpha} = \frac{2}{c^2} \int \boldsymbol{\nabla}_\xi \Phi \, d\ell$$

where $\boldsymbol{\nabla}_\xi$ is the two-dimensional gradient operator in the lens plane. Show how this bend angle is related to angular coordinates on the sky of images $\boldsymbol{\theta}$ and the undisplaced coordinates $\boldsymbol{\beta}$. Defining the lensing potential $\boldsymbol{\theta} - \boldsymbol{\beta} = \boldsymbol{\nabla}_\theta \psi$, determine the relation of the lensing potential $\psi$ to the 3D gravitational potential $\Phi$. Show that $\psi$ satisfies the 2D Poisson equation $\nabla_\theta^2 \psi = 2\kappa$, where

$$\kappa = \frac{4\pi G}{c^2} \int \frac{D_\ell D_{\ell s}}{D_s} \, \bar{\rho}\, \delta \, d\ell;$$

explain clearly the meaning of all terms in this equation. **[7]**

(c) Writing the lensing Jacobian transformation matrix in terms of the lensing potential and then the reduced shear, describe how gravitational lensing changes the appearance of objects in the image plane. Explain how, in the Born-approximation limit of weak lensing, image distortions can be superimposed, so that the definition of $\kappa$ need not be restricted to a thin screen. **[6]**

(d) Write notes to explain in outline how gravitational lensing may be used as a probe of the properties of dark energy. **[6]**