Advanced Cosmology: Summer 2008

Section A Answer two Questions

A.1.

(a) Starting from the Friedmann equation, show that the Hubble parameter as a function of epoch can be written as

$$H^2(a) = H_0^2 \left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v a^{-3(w+1)} + (1 - \Omega_t) a^{-2} \right),$$

defining carefully all terms that appear. **[6]**

(b) Write down the integral for the relation between comoving distance and redshift, and explain the meaning of the terms 'particle horizon' and 'event horizon'. Give the relation between the current distance to an object at redshift z, the current particle horizon, and the particle horizon at the time when the light we now receive was emitted. **[6]**

(c) The North and South Hubble Deep Fields are two small patches that lie in opposite directions on the sky, and which contain statistically identical galaxy populations. Show that, according to the above model for $H(a)$, there are critical redshifts beyond which galaxies that we can observe in the two Hubble Deep Fields have not established causal contact (a) by the present day; (b) by the time at which the light we now see was emitted. Considering the following table of comoving distances for a flat $\Omega_m = 0.25$ model, estimate these redshifts. **[8]**

z	$D(z)/h^{-1}\,\mathrm{Mpc}$
0.5	1345
1	2385
1.5	3178
2	3795
3	4690
5	5775
10	7051
∞	10666

(d) Discuss this 'horizon problem' and explain how an early vacuum-dominated phase of accelerated expansion permits causal contact to be achieved throughout the universe. Define the equation-of-state parameter w and state the condition on w needed in order to solve the horizon problem. **[5]**

A.2. A flat expanding universe contains a mean density of pressureless matter, ρ_m and a mean density of relativistic matter, ρ_r, with fractional density fluctuations in each component of respectively δ_m and δ_r. The growth of these fluctuations is described by the equations

$$\begin{aligned}\ddot{\delta}_m + 2\frac{\dot{a}}{a}\dot{\delta}_m &= 4\pi G\left(\rho_m\,\delta_m + 2\rho_r\delta_r\right)\\ \ddot{\delta}_r + 2\frac{\dot{a}}{a}\dot{\delta}_r &= 16\pi G\left(\rho_m\,\delta_m + 2\rho_r\delta_r\right)/3.\end{aligned}$$

where $\bar{\rho}_m$ is the mean matter density.

(a) Define the distinction between adiabatic and isocurvature perturbation modes. Prove that, according to the above equations, an adiabatic mode preserves a fixed ratio between δ_m and δ_r at all times. **[6]**

(b) In the limit of short wavelengths, these equations are modified, with relativistic streaming forcing $\delta_r \to 0$. With this modification, the equation for δ_m still applies. Consider the evolution of these matter fluctuations through the era of matter-radiation equality, defining $y(t) = \rho_m/\rho_r$. Show that the growth equation at a time where curvature and vacuum energy can be neglected becomes

$$\delta_m'' + \frac{2+3y}{2y(1+y)}\,\delta_m' - \frac{3}{2y(1+y)}\,\delta_m = 0,$$

where dashes denote d/dy. You may find it helpful to note that $H = \dot{a}/a$ can be written as $H^2 \propto (y^{-3} + y^{-4})$. **[8]**

(c) Show that $\delta_m \propto y + 2/3$ is a solution of this equation, and use this to derive an expression for the dependence on scale factor of gravitational potential perturbations. Give a physical explanation for this behaviour. **[5]**

(d) In terms of this solution, discuss the form of the matter transfer function, explaining the characteristic length scales that it contains, and their dependence on cosmological parameters. **[6]**

A.3.

The photons that constitute the Cosmic Microwave background were last scattered at a mean redshift of 1070, with an rms dispersion in scattering redshift of 80. At this redshift, the relativistic density cannot be completely neglected. In a flat vacuum-dominated universe, a good approximation to the comoving distance to last scattering is $D_{\rm LS} = (2c/H_0)\Omega_m^{-0.4}$.

(a) Calculate the comoving thickness of the last-scattering shell. Assuming that this represents the smallest length-scale of surviving CMB temperature fluctuations, estimate the angular scale below which the CMB temperature would appear uniform. **[6]**

(b) Derive an expression for the comoving horizon size at last scattering, and show that the angle that this length subtends today, $\theta_{\rm H-LS}$, is of order 1 degree.

Explain without detailed calculation how this result would differ in an open universe with zero vacuum density. [**8**]

(c) Explain in outline why this 1-degree scale is expected to be the dominant scale in the pattern of CMB temperature anisotropies. [**5**]

(d) Adopting $\Omega_m = 0.25$ and $h = 0.73$, calculate the response of the horizon angle to small changes in these parameters, in the approximate form $\theta_{\mathrm{H-LS}} \propto \Omega_m^a h^b$. Explain how this result combines with information from large-scale structure to allow both Ω_m and h to be measured. [**6**]