MPhys Advanced Cosmology 2009/2010

Examinable material

The following equations should be considered examinable in the sense that you may be expected to remember them and be able to use them. Long derivations are less likely to be required – and certainly will not be requested unless the purpose of a question is to lead you with hints through the steps of such a derivation.

RW metric:

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left[dr^2 + S_k^2(r)\, d\psi^2\right],$$

where the function $S_k(r)$ is

$$S_k(r) \equiv \begin{cases} \sin r & (k=1) \\ \sinh r & (k=-1) \\ r & (k=0). \end{cases}$$

Hubble parameter:

$$H(t) = \dot{R}(t)/R(t).$$

Dimensionless (current) Hubble parameter:

$$h \equiv \frac{H_0}{100\,\mathrm{km\,s^{-1}Mpc^{-1}}},$$

Dimensionless scale factor:

$$a(t) \equiv \frac{R(t)}{R_0} = 1/(1+z),$$

Friedmann equation version 1:

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2.$$

Friedmann equation version 2:

$$\ddot{R} = -4\pi G R(\rho + 3p/c^2)/3.$$

Density parameter:

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}.$$

Equation of state:

$$w \equiv p/\rho c^2 \Rightarrow \rho \propto a^{-3(w+1)}.$$

Time-dependent Hubble parameter:

$$H^2(a) = H_0^2\left[\Omega_v + \Omega_m a^{-3} + \Omega_r a^{-4} - (\Omega - 1)a^{-2}\right].$$

Flat solutions:

$$a \propto t^{1/2} \quad \text{(radiation)}, \quad t^{2/3} \quad \text{(matter)}.$$

Distance-redshift relation:

$$R_0 r = \int \frac{c}{H(z)}\, dz.$$

Current horizon for flat vacuum + matter model (comoving distance to $z = \infty$):

$$R_0 r_{\mathrm{H}} \simeq \frac{2c}{H_0} \Omega_m^{-0.4}.$$

Photon number density and entropy density:

$$n \propto s \propto T^3.$$

Redshift of matter–radiation equality:

$$1 + z_{\mathrm{eq}} = 24\,074\, \Omega h^2.$$

Boltzmann equation for relics:

$$\dot{n} + 3Hn = -\langle \sigma v \rangle (n^2 - n_{\mathrm{T}}^2).$$

Optical depth to last scattering:

$$\tau(z) \simeq \left(\frac{1+z}{1080} \right)^{13} = 1 \text{ at } z = 1100 \pm 80.$$

Planck mass:

$$m_{\mathrm{P}} \equiv \sqrt{\frac{\hbar c}{G}} \simeq 10^{19} \mathrm{GeV}.$$

Lagrangian density for real scalar field with mass term:

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu \phi \partial_\mu \phi - \mu^2 \phi^2).$$

Energy density and pressure for homogeneous field:

$$\rho = \tfrac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$p = \tfrac{1}{2}\dot{\phi}^2 - V(\phi).$$

Equation of motion for homogeneous field:

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0.$$

Dimensionless fluctuation amplitude

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}}.$$

Comoving Fourier expansion:

$$\delta(\mathbf{x}) = \sum \delta_k e^{-i\mathbf{k}\cdot\mathbf{x}} = \left(\frac{L}{2\pi} \right)^3 \int_{-\infty}^{\infty} \delta_k e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3k.$$

Power spectrum:

$$\langle \delta^2 \rangle = \sum |\delta_k|^2.$$

Dimensionless Power spectrum:

$$\Delta^2(k) \equiv \frac{L^3}{(2\pi)^3} \, 4\pi k^3 \, |\delta_k|^2.$$

Poisson's equation in Fourier space:

$$a^{-2}\nabla^2\Phi = -(k^2/a^2)\Phi = 4\pi G\bar{\rho}\,\delta.$$

Newtonian-gauge metric:

$$d\tau^2 = (1+2\Phi)dt^2 - (1-2\Phi)\gamma_{ij}\,dx^i\,dx^j.$$

Fluctuation in inflation field:

$$\delta\phi = H/2\pi.$$

Horizon-scale amplitude:

$$\delta_{\rm H} \equiv \sqrt{\Delta_\Phi^2} \simeq H\;\delta t = \frac{H}{\dot{\phi}}\,\delta\phi = \frac{H^2}{2\pi\,\dot{\phi}}.$$

Force equation for comoving peculiar velocity:

$$\dot{\mathbf{u}} + 2(\dot{a}/a)\mathbf{u} = \mathbf{g}/a.$$

Continuity equation:

$$\frac{d}{dt}\delta = -(1+\delta)\boldsymbol{\nabla}\cdot\mathbf{u}.$$

Linearised continuity equation:

$$\dot{\delta} = -\boldsymbol{\nabla}\cdot\mathbf{u}.$$

Linear equation for matter-dominated density fluctuations, including pressure:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \delta\big(4\pi G\rho_0 - c_s^2 k^2/a^2\big).$$

Growing mode in matter-dominated regime (neglecting curvature):

$$\delta(t) \propto t^{2/3}.$$

Growing mode in radiation-dominated regime:

$$\delta(t) \propto t.$$

Transfer function:

$$\Delta^2(k) \propto k^{3+n}\,T_k^2.$$

Horizon size at matter-radiation equality (break scale in T_k):

$$D_{\rm H}(z_{\rm eq}) = 16\,(\Omega_m h^2)^{-1}{\rm Mpc}.$$

Displacement field:

$$\mathbf{x}(t) = a(t)\,(\mathbf{q} + \mathbf{f}(\mathbf{q},t))\,.$$

Density field:

$$\rho\,/\,\bar{\rho} = \left|\frac{\partial\mathbf{q}}{\partial\mathbf{x}/a}\right| \simeq 1 - \boldsymbol{\nabla}\cdot\mathbf{f} \quad (f\to 0).$$

Spherical model criteria for virialization:

$$\rho/\bar{\rho} \simeq 178; \quad \delta_{\rm lin} \simeq 1.686.$$

Relation of mass function and collapse fraction:

$$Mf(M)/\rho_0 = |dF_{\rm coll}/dM|.$$

Dimensionless surface mass density, or convergence

$$\kappa(\boldsymbol{\theta}) = \frac{\Sigma({\rm D_d}\,\boldsymbol{\theta})}{\Sigma_{\rm cr}}.$$

Lensing Jacobian

$$A_{ij} = \left(\frac{\partial(\beta_i)}{\partial(\theta_j)}\right)_{ij} = \delta_{ij} - \frac{\partial^2\psi}{\partial\theta_i\partial\theta_j}.$$

Relationship between the lensing potential, convergence and shear

$$\kappa \equiv (\psi_{11} + \psi_{22})/2$$
$$\gamma_1 \equiv (\psi_{11} - \psi_{22})/2$$
$$\gamma_2 \equiv \psi_{12},$$

Simple relationship between observed galaxy ellipticity and the reduced shear $g = \gamma/(1-\kappa)$

$$e^s = e - g\,.$$

CMB spherical harmonic expansion:

$$\frac{\delta T}{T}(\hat{\mathbf{q}}) = \sum a_\ell^m Y_{\ell m}(\hat{\mathbf{q}}).$$

CMB dimensionless power per log ℓ:

$$\mathcal{T}^2(\ell) = \ell(\ell+1)C_\ell/2\pi.$$

Gravitational (Sachs–Wolfe) perturbations:

$$\frac{\delta T}{T} = \frac{1}{3}(\Phi/c^2).$$

Intrinsic (adiabatic) perturbations:

$$\frac{\delta T}{T} = \frac{\delta(z_{{\rm L}S})}{3},$$

Velocity (Doppler) perturbations:

$$\frac{\delta T}{T} = \frac{\delta\mathbf{v}\cdot\hat{\mathbf{r}}}{c}.$$

ISW effect:

$$\frac{\delta T}{T} = 2\int \frac{\dot{\Phi}}{c^2}\,dt.$$

Horizon size at last scattering:

$$D_{\rm H}^{{\rm L}S} = 184\,(\Omega_m h^2)^{-1/2}{\rm Mpc}.$$

Reionization optical depth:

$$\tau = \int \sigma_{\rm T}\,n_e\,d\ell_{\rm prop}.$$

Friedmann equation using CMB observables ($\omega \equiv \Omega h^2$):

$$h^2 = \omega_m + \omega_v + \omega_k.$$

Amplitude of tensor perturbations:

$$\left(\frac{\delta T}{T}\right)_{\rm GW} \sim h_{\rm rms} \sim H/m_{\rm P} \sim V^{1/2}/m_{\rm P}^2.$$