So far, we have concentrated on describing perturbations in the matter density, and will go on to discuss ways in which these may be observed. But first, we should put in place the corresponding machinery for the fluctuations in the radiation density. These can be observed directly in terms of fluctuations in the temperature of the CMB, which relate to the density fluctuation field at $z \simeq 1100$. We therefore have the chance to observe both current cosmic structure and its early seeds. By putting the two together and requiring consistency, the cosmological model can be pinned down with amazing precision.

### 8.1 Anisotropy mechanisms

Fluctuations in the 2D temperature perturbation field are treated similarly to density fluctuations, except that the field is expanded in spherical harmonics, so modes of different scales are labelled by multipole number, $\ell$:

$$
\frac{\delta T}{T}(\hat{q}) = \sum a^m_\ell Y^m_\ell(\hat{q}),
$$

where $\hat{q}$ is a unit vector that specifies direction on the sky. The spherical harmonics satisfy the orthonormality relation

$$
\int Y^m_\ell(\hat{q}) Y^*_{m'}(\hat{q}) d^2q = \delta_{\ell\ell'}\delta_{mm'},
$$

so the variance in temperature averaged over the sky is

$$
\langle \left(\frac{\delta T}{T}\right)^2 \rangle = \frac{1}{4\pi} \sum_{\ell,m} |a^m_\ell|^2 = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) C_\ell
$$

The spherical harmonics are familiar as the eigenfunctions of the angular part of $\nabla^2$, and there are $2\ell + 1$ modes of given $\ell$, hence the notation for the angular power spectrum, $C_\ell$. For $\ell \gg 1$, the spherical harmonics become equivalent to Fourier modes, in which the angular wavenumber is $\ell$; therefore one can associate a ‘wavelength’ $2\pi/\ell$ with each mode.

Once again, it is common to define a ‘power per octave’ measure for the temperature fluctuations:

$$
T^2(\ell) = \ell(\ell + 1)C_\ell/2\pi
$$

(although shouldn’t $\ell(\ell+1)$ be $\ell(\ell+1/2)$? – see later). Note that $T^2(\ell)$ is a power per ln $\ell$; the modern trend is often to plot CMB fluctuations with a linear scale for $\ell$ – in which case one should really use $T^2(\ell)/\ell$.

We now list the mechanisms that cause **primary anisotropies** in the CMB (as opposed to **secondary anisotropies**, which are generated by scattering along the line of sight). There are three basic primary effects, illustrated in figure 17, which are important on respectively large, intermediate and small angular scales:
Figure 17. Illustrating the physical mechanisms that cause CMB anisotropies. The shaded arc on the right represents the last-scattering shell; an inhomogeneity on this shell affects the CMB through its potential, adiabatic and Doppler perturbations. Further perturbations are added along the line of sight by time-varying potentials (Rees–Sciama effect) and by electron scattering from hot gas (Sunyaev–Zeldovich effect). The density field at last scattering can be Fourier analysed into modes of wavevector $k$. These spatial perturbation modes have a contribution that is in general damped by averaging over the shell of last scattering. Short-wavelength modes are more heavily affected (i) because more of them fit inside the scattering shell, and (ii) because their wavevectors point more nearly radially for a given projected wavelength.

(1) Gravitational (Sachs–Wolfe) perturbations. Photons from high-density regions at last scattering have to climb out of potential wells, and are thus redshifted:

$$\frac{\delta T}{T} = \frac{1}{3}(\Phi/c^2).$$

(241)

The factor $1/3$ is a surprise, which arises because $\Phi$ has two effects: (i) it redshifts the photons we see, so that an overdensity cools the background as the photons climb out, $\delta T/T = \Phi/c^2$; (ii) it causes time dilation at the last-scattering surface, so that we seem to be looking at a younger (and hence hotter) universe where there is an overdensity. The time dilation is $\delta t/t = \Phi/c^2$; since the time dependence of the scale factor is $a \propto t^{2/3}$ and $T \propto 1/a$, this produces the counterterm $\delta T/T = -(2/3)\Phi/c^2$.

(2) Intrinsic (adiabatic) perturbations. In high-density regions, the coupling of matter and radiation can compress the radiation also, giving a higher temperature:

$$\frac{\delta T}{T} = \frac{\delta(z_{LS})}{3},$$

(242)

(3) Velocity (Doppler) perturbations. The plasma has a non-zero velocity at recombination, which leads to Doppler shifts in frequency and hence brightness temperature:

$$\frac{\delta T}{T} = \frac{\delta v \cdot \hat{r}}{c}. \quad (243)$$
To the above list should be added ‘tensor modes’: anisotropies
due to a background of primordial gravitational waves, potentially
generated during an inflationary era (see below).

There are in addition effects generated along the line of sight.
One important effect is the integrated Sachs-Wolfe effect (ISW
effect), which arises when the potential perturbations evolve:

$$\frac{\delta T}{T} = \frac{1}{c^2} \int \dot{\Psi} + \dot{\Phi} \, dt.$$  \hfill (244)

In the usual $\Psi = \Phi$ limit, this is twice as large as one might have
expected from Newtonian intuition. This factor 2 thus has an origin
that is similar to the factor 2 for relativistic light deflection (where the
one-line argument is that the gravitational potential modifies both the
time and space parts of the metric, and each contribute equally to the
effective change in the coordinate speed of light). But the ISW effect
is a little more subtle, and we shall just accept the result as intuitively
plausible. As we have seen, the potential $\Phi$ stays constant in the linear
regime during the matter-dominated era, as long as $\Omega_m \simeq 1$, so the
source term for the ISW effect vanishes for much of the universe’s
history. The ISW effect then becomes only important quite near to
the last scattering redshift (because radiation is still important) and
al low $z$ (because of $\Lambda$).

Other foreground effects are to do with the development of
nonlinear structure, and are mainly on small scales (principally the
Sunyaev–Zeldovich effect from IGM Comptonization). The exception
is the effect of reionization; to a good approximation, this merely
damps the fluctuations on all scales:

$$\frac{\delta T}{T} \rightarrow \frac{\delta T}{T} \exp -\tau,$$ \hfill (245)

where the optical depth must exceed $\tau \simeq 0.04$, based on the highest-
redshift quasars and the BBN baryon density. As we will see later,
CMB polarization data have detected a signature consistent with
$\tau = 0.1 \pm 0.03$, implying reionization at $z \simeq 10$.

## 8.2 Power spectrum

We now need to see how the angular power spectrum of the CMB arises
from the implementation of these effects. The physical separation
we have made is useful for insight, although it is not exactly how
things are calculated in practice. We have not been able to spend
time going into the detailed formalism used on CMB anisotropies,
and the details will have to be omitted here – although the actual
equations to be integrated are not enormously complicated. For the
present purpose, we will make a few comments about why the exact
approach is complicated, and then retreat to a simpler approximate
treatment.

The natural approach is to start in Fourier space and consider a
density fluctuation of given wavevector $k$; if we can work out how this
appears as an induced temperature fluctuation on the CMB sky, then
the problem can be solved by superposition. The wavevector $k$ sets
a natural polar axis, and the temperature anisotropy corresponds to
knowing the photon phase-space distribution at our location in space
(i.e. the distribution of the photons in energy and as a function of
angle with respect to $k$). Evolving this function is hard principally
because of the coupling between radiation and matter, which is by Thomson scattering. Scattering a beam of photons that come from a given direction will tend to push the electron in the opposite direction, so a net force requires an anisotropic the photon distribution. In fact, it is clear that the force must be proportional to the dipole moment of the distribution function, and this is obviously a problem: it couples the evolution of the number of photons travelling at a given direction with a knowledge of the whole distribution. Mathematically, we have an integro-differential equation.

In practice, rather than trying to solve numerically for the photon distribution function (normally denoted by $\Theta$), we can carry out a multipole transform to work with $\Theta_\ell$. The integro-differential equation then becomes a set of equations that couple different $\ell$ values. These have to be solved as a large set of equations (we will see that the CMB power spectrum contains signal at least to $\ell \gtrsim 1000$), and when this is done we still have to integrate over $k$ space. It took many years to solve this numerical challenge, and even then the computations were very slow. But a key event in cosmology was the 1996 release of CMBFAST, a public Boltzmann code that allowed computation of the CMB angular power spectrum sufficiently rapidly that a large range of models could be investigated by non-specialists.

**TIGHT-COUPLING PROJECTION APPROACH** An alternative approximate method is to imagine that the temperature anisotropies exist as a 3D spatial field. The last-scattering surface can be envisaged as a slice through this field, so the angular properties are really just a question of understanding the projection that is involved. This works reasonably well in the **tight coupling** limit where photons and baryons are a single fluid – but this is of course breaking down at last scattering, where the photon mean free path is becoming large.

The projection is easily performed in the **flat-sky approximation**, where we ignore the curvature of the celestial sphere. The angular wavenumber is then just $\ell = KD_H$, where $D_H$ is the distance to the last-scattering surface and $K$ is a 2D transverse physical wavenumber ($K^2 = k_x^2 + k_y^2$). The relation between 3D and 2D power spectra is easily derived: we just add up the power along the unused axis, $k_z$:

$$P_{2D}(k_x, k_y) = \sum_{k_z} P_{3D}(k_x, k_y, k_z) = \frac{L}{2\pi} \int_{-\infty}^{\infty} P_{3D}(k) \, dk_z. \quad (246)$$

In terms of dimensionless power, this is

$$\Delta_{2D}^2(K) = \left( \frac{L}{2\pi} \right) ^2 2\pi K^2 P_{2D}(K) = K^2 \int_0^{\infty} \Delta_{3D}^2(k) \, dk_z / k^3, \quad (247)$$

where $k^2 = K^2 + k_z^2$. The 2D spectrum is thus a smeared version of the 3D one, but the relation is pleasingly simple for a scale-invariant spectrum in which $\Delta_{3D}^2(k)$ is a constant:

$$\Delta_{2D}^2(K) = \Delta_{3D}^2. \quad (248)$$

The important application of this is to the Sachs-Wolfe effect, where the 3D dimensionless spectrum of interest is that of the potential, $\Delta_{3D}^2 = \delta_H^2$. This shows that the angular spectrum of the CMB should have a flat portion at low $\ell$ that measures directly the metric fluctuations.
This is the signature that formed the first detection of CMB anisotropies – by COBE in 1992; we will see below that this corresponds to

\[ \delta_H \simeq 3 \times 10^{-5}. \]  

(249)

This immediately determines the large-scale matter power spectrum in the universe today. We know from Poisson’s equation that the relation between potential and density power spectra at scale factor \( a \) is

\[ \Delta_\Phi^2 = (4\pi G \rho_m a^2 / k^2)^2 \Delta^2(a) \equiv \delta_H^2. \]  

(250)

Converting to the present, \( \Delta^2 = a^{-2} \Delta^2(a) f(\Omega_m)^2, \) and we get

\[ \Delta^2 = (4/9) \delta_H^2 \left( \frac{ck}{H_0} \right)^4 \Omega_m^{-2} f(\Omega_m)^2 \]  

(251)

(where \( f(\Omega_m) \simeq \Omega_m^{0.23} \) for a flat universe, is the growth suppression factor). This expression is modified on small scales by the transfer function, but it shows how mass fluctuations today can be deduced from CMB anisotropies. As an aside, a more informal argument in the opposite direction is to say that we can estimate the depth of potential wells today:

\[ v^2 \sim \frac{GM}{r} \Rightarrow \frac{\Phi}{c^2} \sim \frac{v^2}{c^2}, \]  

(252)

so the potential well of the richest clusters with velocity dispersion \( \sim 1000 \text{ km s}^{-1} \) is of order \( 10^{-5} \) deep. It is therefore no surprise to see this level of fluctuation on the CMB sky.

Finally, it is also possible with some effort to calculate the full spherical-harmonic spectrum from the 3D spatial spectrum. For a scale-invariant spectrum, the result is

\[ C_\ell = \frac{6}{\ell(\ell + 1)} C_2, \]  

(253)

which is why the broad-band measure of the ‘power per log \( \ell \)’ is defined as

\[ T^2(\ell) = \frac{\ell(\ell + 1)}{2\pi} C_\ell. \]  

(254)

Finally, a word about units. The temperature fluctuation \( \Delta T/T \) is dimensionless, but anisotropy experiments generally measure \( \Delta T \) directly, independent of the mean temperature. It is therefore common practice to quote \( T^2 \) in units of \( (\mu\text{K})^2 \).

**Characteristic Scales.** We now want to look at the smaller-scale features of the CMB. The current data are contrasted with some CDM models in figure 18. The key feature that is picked out is the dominant peak at \( \ell \simeq 220 \), together with harmonics of this scale at higher \( \ell \). How can these features be understood?

The main point to appreciate is that the gravitational effects are the ones that dominate on large angular scales. This is easily seen
Figure 18. Angular power spectra $T^2(\ell) = \ell(\ell + 1)C_\ell / 2\pi$ for the CMB, plotted against angular wavenumber $\ell$ in radians$^{-1}$. For references to the experimental data, see Spergel et al. (2006). The two lines show model predictions for adiabatic scale-invariant CDM fluctuations, calculated using the \textsc{cmbfast} package (Seljak & Zaldarriaga 1996). These have $(n, \Omega_m, \Omega_b, h) = (1, 0.3, 0.05, 0.65)$ and have respectively $\Omega_v = 1 - \Omega_m$ (‘flat’) and $\Omega_v = 0$ (‘open’). The main effect is that open models shift the peaks to the right, as discussed in the text.

by contrasting the temperature perturbations from the gravitational and adiabatic perturbations:

$$\frac{\delta T}{T} \simeq \frac{1}{3} \frac{\Phi}{c^2} \quad \text{(gravity)}; \quad \frac{\delta T}{T} \simeq \frac{1}{3} \frac{\delta \rho}{\rho} \quad \text{(adiabatic).}$$  \hspace{1cm} (255)

Poisson’s equation says $\nabla^2 \Phi = -k^2 \Phi = 4\pi G \rho (\delta \rho / \rho)$, so there is a critical (proper) wavenumber where these two effects are equal: $k_{\text{crit}}^2 \sim G \rho / c^2$. The age of the universe is always $t \sim (G \rho)^{-1/2}$, so this says that

$$k_{\text{crit}} \sim (ct)^{-1}. \hspace{1cm} (256)$$

In other words, perturbations with wavelengths above the horizon size at last scattering generate $\delta T / T$ via gravitational redshift, but on smaller scales it is adiabatic perturbations that matter.

The significance of the main \textbf{acoustic peak} is therefore that it picks out the (sound) horizon at last scattering. The redshift of last scattering is almost independent of cosmological parameters at
\[ z_{\text{LS}} \simeq 1100, \text{ as we have seen. If we assume that the universe is matter} \]
\[ \text{dominated at last scattering, the horizon size is} \]
\[ D_{\text{H}}^{\text{LS}} = 184 (\Omega_m h^2)^{-1/2} \text{Mpc.} \]  

The angle this subtends is given by dividing by the current size of the horizon (strictly, the comoving angular-diameter distance to \( z_{\text{LS}} \)). Again, for a matter-dominated model with \( \Lambda = 0 \), this is
\[ D_{\text{H}} = 6000 \Omega_m^{-1} h^{-1} \text{Mpc} \Rightarrow \theta_{\text{H}} = D_{\text{H}}^{\text{LS}} / D_{\text{H}} = 1.8 \Omega_m^{0.5} \text{degrees.} \]

Figure 18 shows that heavily open universes thus yield a main CMB peak at scales much smaller than the observed \( \ell \simeq 220 \), and these can be ruled out. Indeed, open models were disfavoured for this reason long before any useful data existed near the peak, simply because of strict upper limits at \( \ell \simeq 1500 \) (Bond & Efstathiou 1984). In contrast, a flat vacuum-dominated universe has \( D_{\text{H}} \simeq 6000 \Omega_m^{-0.4} h^{-1} \text{Mpc} \), so the peak is predicted at \( \ell \simeq 2\pi/(184/6000) \simeq 200 \) almost independent of parameters. These expression lie behind the common statement that the CMB data require a flat universe – although it turns out that large degrees of spatial curvature and \( \Lambda \) can also match the CMB well.

The second dominant scale is imposed by the fact that the last-scattering surface is fuzzy – with a width in redshift of about \( \delta z = 80 \). This imposes a radial smearing over scales \( \sigma_r = 7(\Omega_m h^2)^{-1/2} \text{ Mpc} \). This subtends an angle
\[ \theta_r \simeq 4 \text{ arcmin}, \]  

for flat models. This is partly responsible for the fall in power at high \( \ell \) (Silk damping also contributes). Finally, a characteristic scale in many density power spectra is set by the horizon at \( z_{\text{eq}} \). This is \( 16(\Omega h^2)^{-1} \text{ Mpc} \) and subtends a similar angle to \( \theta_r \).

REIONIZATION As mentioned previously, it is plausible that energy output from young stars and AGN at high redshift can reionize the intergalactic medium. Certainly, we know empirically from the lack of Gunn–Peterson neutral hydrogen absorption in quasars that such reheating did occur, and at a redshift in excess of 6. The consequences for the microwave background of this reionization depend on the Thomson-scattering optical depth:

\[ \tau = \int \sigma_T n_e \, dt_{\text{prop}} = \int \sigma_T n_e \frac{c}{H_0} \frac{dz}{(1+z)\sqrt{1-\Omega_m + \Omega_m (1+z)^3}} \]  

(for a flat model). If we re-express the electron number density in terms of the baryon density parameter as
\[ n_e = \Omega_b \frac{3H_0^2}{8\pi G \mu m_p} (1+z)^3, \]

where the parameter \( \mu \) is approximately 1.143 for a gas of 25% helium by mass, and do the integral over redshift, we get

\[ \tau = 0.04h \frac{\Omega_b}{\Omega_m} \left[ \sqrt{1 + \Omega_m z(3 + 3z + z^2)} - 1 \right] \simeq 0.04h \frac{\Omega_b}{\Omega_m^{1/2}} z^{3/2}. \]  

Predictions from CDM galaxy formation models tend to predict a reheating redshift between 10 and 15, thus \( \tau \) between 0.1 and 0.2 for
standard parameters. The main effect of this scattering is to damp the CMB fluctuations by a factor \( \exp(-\tau) \), but this does not apply to the largest-scale angular fluctuations. To see this, think backwards: where could a set of photons scattered at \( z \) have come from? If they are scattered by an angle of order unity, they can be separated at the last-scattering surface by at most the distance from \( z \) to \( z_{LS} \) – which is almost exactly the horizon size at \( z \). The critical angle is thus the angle subtended today by the horizon size at the reheating time; for a flat model, this is approximately \( \frac{1}{2} \) radians, so modes with \( \ell < \frac{1}{2} z \) are unaffected. This turns out to be a critical factor in changing the apparent shape of the CMB power spectrum.

9 CMB anisotropies – II

Having given an outline of the physical mechanisms that contribute to the CMB anisotropies, we now examine how the CMB is used in conjunction with other probes to pin down the cosmological model.

9.1 Geometrical degeneracies

The first point to appreciate is that the appearance of the CMB only depends on a restricted set of combinations of cosmological parameters, and we cannot measure all parameters at once from the CMB alone. For a given primordial spectrum (i.e. given \( n_s \)), the CMB temperature power spectrum depends on the physical densities \( \omega_m \equiv \Omega_m h^2 \), \( \omega_b \equiv \Omega_b h^2 \). Now, it is possible to vary both \( \Omega_v \) and the curvature to keep a fixed value of the angular size distance to last scattering, so that the resulting angular CMB pattern will be invariant.

The usual expression for the comoving angular-diameter distance is

\[
R_0 S_k(r) = \frac{c}{H_0} |1 - \Omega|^{-1/2} \times \sqrt{\left[ \int_0^z \frac{|1 - \Omega|^{1/2} \, dz'}{\sqrt{\omega_k(1 + z')^2 + \omega_v + \omega_m(1 + z')^3}} \right]}, \tag{263}
\]

where \( \Omega = \Omega_m + \Omega_v \). Defining \( \omega_k \equiv \Omega_k h^2 \), this can be rewritten in a way that has no explicit \( h \) dependence:

\[
R_0 S_k(r) = \frac{3000 \text{ Mpc}}{|\omega_k|^{1/2}} S_k \left[ \int_0^z \frac{|\omega_k|^{1/2} \, dz'}{\sqrt{\omega_k(1 + z')^2 + \omega_v + \omega_m(1 + z')^3}} \right], \tag{264}
\]

where \( \omega_k \equiv (1 - \Omega_m - \Omega_v) h^2 \). This parameter describes the curvature of the universe, treating it effectively as a physical density that scales as \( \rho \propto a^{-2} \); in the Friedmann equation, curvature cannot be distinguished from another contribution to the density, although clearly the form of the RW metric is able to tell the difference. The \( \omega_k \) notation is therefore slightly misleading.

For fixed \( \omega_m \) and \( \omega_b \), there is therefore a degeneracy between curvature (\( \omega_k \)) and vacuum (\( \omega_v \)): these two parameters can be varied simultaneously to keep the same apparent distance, and hence the same angular structure in the CMB. govern the proportionality between length at last scattering and observed angle. The degeneracy is not exact, and is weakly broken by the Integrated Sachs-Wolfe effect from evolving potentials at very low multipoles, and by second-order effects at high \( \ell \). However, strong breaking of the degeneracy
Figure 19. The CMB geometrical degeneracy, for the fiducial flat vacuum-dominated model. This says that zero vacuum density can be tolerated, provided the ‘curvature density’ is negative – i.e. a closed universe. Since the fiducial value of \( \omega_m \) is 0.133, \( \omega_k = -0.031 \) implies that a Hubble parameter \( h = 0.32 \) would be required in this case. Provided we are convinced that \( h \) must be higher than this, vacuum energy is required.

requires additional information. This could be in the form of external data on the Hubble constant, which is determined implicitly by the relation

\[
h^2 = \omega_m + \omega_v + \omega_k, \tag{265}
\]

so specifying \( h \) in addition to the physical matter density fixes \( \omega_v + \omega_k \) and removes the degeneracy. In practice, the most accurate constraints are obtained by adding independent probes that have sensitivity to the matter density: the supernova Hubble diagram; large-scale structure; gravitational lensing.

HORIZON-ANGLE DEGENERACY For flat models, we therefore see that perfect data on the CMB can in principle determine all the main cosmological parameters. At present, the CMB is not measured perfectly, and there is in any case a fundamental limit to how well this can be done. This comes from cosmic variance: observers at widely spaced locations will see different realizations of the density fluctuation process around them. On the finite sky, there is only a finite number of independent modes, and this limits how well the power spectrum can be measured to a fractional accuracy that is just \( 1/N_{\text{modes}}^{1/2} \). We therefore expect

\[
\delta C_\ell = \left[ C_\ell^2 \frac{2}{2\ell + 1} + \sigma_{\text{noise}}^2 \right]^{1/2} \tag{266}
\]
(the number of independent modes is half of $2\ell + 1$ because the temperature field is real). In practice, the instrumental $\sigma_{\text{noise}}$ rises at high $\ell$, so we expect the large-scale CMB measurements to be cosmic-variance limited. In the 2006 WMAP data, the crossover occurs at about $\ell \approx 400$; with luck, ESA’s Planck mission will raise this to beyond 1000.

Given these limitations, the information we gain from the CMB is dominated by the main acoustic peak at $\ell = 220$, and it is interesting to ask what this tells us. We have argued that the location of this feature marks the angle subtended by the acoustic horizon at last scattering, which has been given as $D^s_{\text{LS}} = 184 (\Omega_m h^2)^{-1/2} \text{Mpc}$. Using the current size of the horizon, the angle subtended in a flat model is

$$D_H = 6000 \Omega_m^{-0.4} h^{-1} \text{Mpc} \implies \theta_H = D^s_{\text{LS}} / D_H \propto \Omega_m^{-0.1},$$

so there is very little dependence of peak location on cosmological parameters.

However, this argument is incomplete in detail because the earlier expression for $D_H(z_{\text{LS}})$ assumes that the universe is completely matter dominated at last scattering, and this is not perfectly true. The comoving sound horizon size at last scattering is defined by

$$D^s_{\text{LS}} = \frac{1}{a_{\text{LS}}} \int_0^{a_{\text{LS}}} \frac{c_s}{(a + a_{\text{eq}})^{1/2}} \text{d}a$$

where vacuum energy is neglected at these high redshifts; the expansion factor $a \equiv (1 + z)^{-1}$ and $a_{\text{LS}}, a_{\text{eq}}$ are the values at last scattering and matter-radiation equality respectively. In practice, $z_{\text{LS}} \simeq 1100$ independent of the matter and baryon densities, and $c_s$ is fixed by $\Omega_b$. Thus the main effect is that $a_{\text{eq}}$ depends on $\Omega_m$. Dividing by $D_H(z = 0)$ therefore gives the angle subtended today by the light horizon as

$$\theta_H \simeq \frac{\Omega_m^{-0.1}}{\sqrt{1 + z_{\text{LS}}}} \left[ \sqrt{\frac{a_{\text{eq}}}{a_{\text{LS}}}} - \sqrt{\frac{a_{\text{eq}}}{a_{\text{LS}}}} \right],$$

where $z_{\text{LS}} = 1100$ and $a_{\text{eq}} = (23900 \omega_m)^{-1}$. This remarkably simple result captures most of the parameter dependence of CMB peak locations within flat $\Lambda$CDM models. Differentiating this equation near a fiducial $\omega_m = 0.13$ gives

$$\frac{\partial \ln \theta_H}{\partial \ln \Omega_m} \bigg|_{\omega_m} = -0.1; \quad \frac{\partial \ln \theta_H}{\partial \ln \omega_m} \bigg|_{\Omega_m} = \frac{1}{2} \left( 1 + \frac{a_{\text{LS}}}{a_{\text{eq}}} \right)^{-1/2} = +0.25,$$

Thus for moderate variations from a ‘fiducial’ model, the CMB peak multipole number scales approximately as $\ell_{\text{peak}} \propto \Omega_m^{-0.15} h^{-0.5}$, i.e. the condition for constant CMB peak location is well approximated as

$$\Omega_m h^{3.3} = \text{constant.}$$

It is now clear how LSS data combines with the CMB: $\Omega_m h$ is the main combination probed by the matter power spectrum so this approximate degeneracy is strongly broken using the combined data.
Figure 20. The location of the principal peak in the CMB power spectrum is largely determined by the combination $\Omega_m h^{3.3}$, representing the scaling of the angular size of the horizon at last scattering. The two other main characteristics are the rise to the peak, and the fall to the second and subsequent maxima. The former (the height of the peak above the Sachs-Wolfe plateau) is influenced by the early-time ISW effect: the change in gravitational potential associated with the transition from radiation domination to matter domination. This is illustrated in the first panel, where we fix $\Omega_b h^2$ and hence the sound speed. For fixed peak location, higher $h$ gives lower matter density, and hence a higher peak from the early-time ISW effect (all models are normalized at $\ell = 20$). The second panel shows the influence of varying the baryon density at constant matter density, where we see that a higher baryon fraction increases the amplitude of the acoustic oscillations. Thus, if we assume flatness, the three observables of the peak location and drops to either side suffice to determine $\Omega_m$, $\Omega_b$ and $h$. Given data over a broader range of $\ell$, the CMB power spectrum is also sensitive to the tilt, $n_s$. 
9.2 Degeneracy breaking with detailed CMB data

Although the main horizon-scale peak in the power spectrum dominates the appearance of the CMB, giving degenerate information about cosmological parameters, the fine detail of the pattern is also important. As the quality of the CMB measurements improve, more information can be extracted, and the parameter degeneracies are increasingly broken by the CMB alone. Regarding the structure around the peak, two physical effects are important in giving this extra information:

(1) **Early ISW.** We have seen that the transition from radiation domination to matter domination occurs only just before last scattering. Although we have proved that potential fluctuations $\Phi$ stay constant during the radiation and matter eras (while vacuum and curvature are negligible), this is not true at the junction, and there is a small change in $\Phi$ during the radiation–matter transition (by a factor 9/10: see chapter 7 of Mukhanov’s book). This introduces an additional ISW effect, which boosts the amplitude of the peak, especially for models with low $\Omega_m h^2$, which brings $z_{eq}$ right down to $z_{LS}$ (see the first panel of figure 20).

(2) **Baryon loading.** If we keep the overall matter density fixed but alter the baryon fraction, the sound speed at last scattering changes. This has the effect of making a change in the amplitude of the acoustic oscillations beyond the first peak: the drop to the second peak is more pronounced if the baryon fraction is high (see the second panel of figure 20).

Overall, the kind of precision data now delivered by WMAP allows these effects to be measured, and the degeneracy between $\Omega_m$, $\Omega_b$ and $h$ broken without external data.

9.3 Tensor modes

All of our discussion to date applies to models in which scalar modes dominate. But we know that gravity-wave metric perturbations in the form of a traceless symmetric tensor $h^{\mu\nu}$ are also possible, and that inflation predicts that a background of such waves is generated, with amplitude

$$h_{\text{rms}} \sim H_{\text{inflation}}/m_p.$$  \hspace{1cm} (272)

These tensor metric distortions are observable via the large-scale CMB anisotropies, where the tensor modes produce a spectrum with the same scale dependence as the Sachs–Wolfe gravitational redshift from scalar metric perturbations. In the scalar case, we have $\delta T/T \sim \phi/3c^2$, i.e. of order the Newtonian metric perturbation; similarly, the tensor effect is

$$\left(\frac{\delta T}{T}\right)_{GW} \sim h_{\text{rms}}.$$  \hspace{1cm} (273)

Could the large-scale CMB anisotropies actually be tensor modes? This would be tremendously exciting, since it would be a direct window into the inflationary era. The Hubble parameter in inflation is $H^2 = \frac{8\pi G \rho}{3} \sim \frac{V(\phi)}{m_p^2}$, so that

$$\left(\frac{\delta T}{T}\right)_{GW} \sim h_{\text{rms}} \sim \frac{H}{m_p} \sim \frac{V^{1/2}}{m_p^2}.$$  \hspace{1cm} (274)
A measurement of the tensor modes in the CMB would therefore tell us directly the energy scale of inflation: \( E_{\text{inflation}} \sim V^{1/4} \). This is more direct than the scalar signature, which was

\[
\delta_H \sim \frac{V^{1/2}}{m_p^2 \epsilon^{1/2}}, \tag{275}
\]

where \( \epsilon \) is the principal slow-roll parameter (dimensionless version of the gradient-squared of the potential).

From these relations, we can see that the tensor-to-scalar ratio in the large-scale CMB power spectra just depends on \( \epsilon \):

\[
r \equiv \frac{T_\ell^2}{T_s^2} = 16\epsilon \tag{276}
\]

(putting in the factor of 16 from an exact analysis). We have argued that \( \epsilon \) cannot be too small if inflation is to end, so significant tensor contributions to the CMB anisotropy are a clear prediction. As a concrete example, consider power-law inflation with \( a \propto t^p \), where we showed that \( \epsilon = \eta/2 = 1/p \). In this case,

\[
r = 8(1 - n_s), \tag{277}
\]

so the larger the tilt, the more important the tensors. We will see below that there is fairly strong evidence for a non-zero tilt with \( n_s \simeq 0.96 \), so the simplest expectation would be a tensor contribution of \( r \simeq 0.3 \). Of course, this only applies for a large-field model like power-law inflation; it is quite possible to have a small-field model with \( |\eta| \gg |\epsilon| \), in which case there can be tilt without tensors.

An order unity tensor contribution would imply metric distortions at the level of \( 10^{-5} \), which might sound easy to detect directly. The reason this is not so is that the small-scale tensor fluctuations are reduced today: their energy density (which is \( \propto h^2 \)) redshifts away as \( a^{-4} \) once they enter the horizon. This redshifting produces a break in the spectrum of waves, reminiscent of the matter transfer spectrum, so that the tensor contribution to the CMB declines for \( \ell \gtrsim 100 \). This redshifting means that the present-day metric distortions are more like \( 10^{-27} \) on relevant scales (kHz gravity waves) than the canonical \( 10^{-5} \). Even so, direct detection of these relic gravity waves can be contemplated, but this will be challenging in the extreme. At the current rate of progress in technology, the necessary sensitivity may be achieved around 2050; but the signal may be higher than in the simple models, so one should be open to the possibility of detecting this ultimate probe of the early universe.