ISOCURVATURE MODES Since there are two degrees of freedom in the matter-radiation perturbation, there must be a second independent perturbation mode to complement the adiabatic solution. This clearly must correspond to a perturbation in the entropy. If we define $S = T^3/\rho_m$, then

$$\delta \ln S = \tfrac{3}{4}\delta_r - \delta_m. \tag{237}$$

In the case of adiabatic initial conditions, S is not perturbed, but the overall energy density fluctuates. The opposite extreme, therefore, is to keep the energy density constant but allow the entropy to fluctuate. We therefore consider constant density at some initial time:

$$\rho_m^i \delta_m^i = -\rho_r^i \delta_r^i. \tag{238}$$

Because spacetime curvature depends on the overall perturbation to the matter density, these initial conditions are known as the **isocurvature mode**. Normally, we will imagine that whatever may generate such a fluctuation in the equation of state happens at early times where things are heavily radiation dominated. In this case, the initial conditions are effectively **isothermal**:

$$\delta_r^i \to 0. \tag{239}$$

The treatment of isothermal perturbations caused some confusion in the older literature. They are not a pure mode – i.e. isothermal perturbations do not stay isothermal, nor indeed does the overall matter density always obey the isocurvature initial condition. This is readily seen by appreciating that (on large scales, where we ignore pressure) causality means that the entropy perturbation must be preserved. Thus, whatever initial conditions we choose for δ_r and δ_m, any subsequent changes to matter and radiation on large scales must be adiabatic. If the initial conditions are effectively isothermal, then

$$\delta_r = \frac{4}{3}(\delta_m - \delta_i), \tag{240}$$

where δ_i is the initial value of δ_m.

Subsequent evolution attempts to preserve the initial constant density by making the matter perturbations decrease while the amplitude of δ_r increases (this $\delta\rho = 0$ condition is not satisfied in detail, but we will skip the full solution here). At late times, $\delta_m \to 0$, while $\delta_r \to -4\delta_i/3$. Hence, as the universe becomes strongly matter dominated, the entropy perturbation becomes carried entirely by the photons. This leads to an increased amplitude of microwave-background anisotropies in isocurvature models, which is one reason why such models are not popular.

It should now be clear why the isothermal perturbation is not a proper mode. At a general time, it corresponds to a mixture of adiabatic and isocurvature perturbations and so cannot stay isothermal. Similarly, in the **curvaton** model, a scalar field with inflationary fluctuations decays at late time to produce a fluctuating radiation field with $\delta_m = 0$. This case yields a mixture of adiabatic and isocurvature modes, such that the temperature fluctuations are comparable at matter-radiation equality. As with pure isocurvature fluctuations, this can be rejected with current data. But more complex models with a small isocurvature fraction are always permitted; one of the tasks of modern cosmology is to improve the limits on such mixtures to the point where the initial conditions are proved to be in effect perfectly adiabatic.

BARYONS AND DARK MATTER This case is simpler, because both components have the same equation of state:

$$L \begin{pmatrix} \delta_b \\ \delta_d \end{pmatrix} = \frac{4\pi G\rho}{\Omega} \begin{pmatrix} \Omega_b & \Omega_d \\ \Omega_b & \Omega_d \end{pmatrix} \begin{pmatrix} \delta_b \\ \delta_d \end{pmatrix}. \tag{241}$$

Both eigenvectors are time independent: $(1,1)$ and $(\Omega_d, -\Omega_b)$. The time dependence of these modes is easy to see for an $\Omega = 1$ matter-dominated universe: if we try $\delta \propto t^n$, then we obtain respectively $n = 2/3$ or -1 and $n = 0$ or $-1/3$ for the two modes. Hence, if we set up a perturbation with $\delta_b = 0$, this mixture of the eigenstates will quickly evolve to be dominated by the fastest-growing mode with $\delta_b = \delta_d$: the baryonic matter falls into the dark potential wells.

We can see that a separation of dark matter and baryons is inevitable by looking at the solution on small scales, where the effect of pressure will prevent the baryons from continuing to follow the dark matter. We can analyse this by writing down the coupled equation for the baryons, but now adding in the pressure term (sticking to the matter-dominated era, to keep things simple):

$$L \begin{pmatrix} \delta_b \\ \delta_d \end{pmatrix} = \frac{4\pi G\rho}{\Omega} \begin{pmatrix} \Omega_b - \kappa^2 & \Omega_d \\ \Omega_b & \Omega_d \end{pmatrix} \begin{pmatrix} \delta_b \\ \delta_d \end{pmatrix}, \tag{242}$$

where $\kappa \equiv k/k_{\rm J}$. This is the quadratic term we saw previously in analysing the Jeans length. In the special case $\Omega_b \to 0$ and $\kappa =$ constant, a solution is clearly

$$\delta_b = \frac{\delta_d}{1+\kappa^2}, \tag{243}$$

and this illustrates the general fact that pressure holds back the growth of baryons during the matter-dominated period.

In fact, such a separation of baryons and dark matter is set up at an earlier time. While the universe is radiation dominated, the baryons and photons act almost as a single fluid, glued together by Thomson scattering (the **tight coupling** approximation). Inside the horizon, the amplitude of perturbations will oscillate with time, effectively as acoustic standing waves. This is in contrast to the behaviour of dark matter, where we have seen that the Mészáros effect causes the amplitude of perturbations in the collisionless component to remain frozen. In addition, the slightly imperfect nature of the matter-radiation fluid causes the oscillation amplitude to undergo viscous damping (**Silk damping**). Thus, baryons end up decoupled from the dark matter fluctuations on scales up to the horizon size at matter-radiation equality, and only start to catch up once the dark matter comes to dominate. The process only properly goes to completion once the Jeans length becomes negligible. This only really happens at the point of **thermal decoupling** around redshift 20. Until this point, Compton scattering is sufficiently rapid that it keeps the primordial plasma heated to the CMB temperature. Only at lower redshifts does the gas cool with the $T_{\rm gas} \propto 1/a^2$ law expected for a normal monatomic gas. The first phases of structure formation are therefore complicated by the fact that dark matter and baryons still have some degree of separation.

6.4 Transfer functions and characteristic scales

The above discussion can be summed up in the from of the linear **transfer function** for density perturbations, where we factor out the long-wavelength growth law from a term that expresses how growth is modulated as a function of wavenumber:

$$\delta(a) \propto g(a) T_k. \tag{244}$$

While curvature is negligible, we have seen that $g(a)$ is proportional to the square of conformal time for adiabatic perturbations. In principle, there is a transfer function for each constituent of the universe, and these evolve with time. As we have discussed, however, the different matter ingredients tend to come together at late times, and the overall transfer function tends to something that is the same for all matter components and which does not change with time for low redshifts. This late-time transfer function is therefore an important tool for cosmologists who want to predict observed properties of density fields in the current universe.

We have discussed the main effects that contribute to the form of the transfer function, but a full calculation is a technical challenge. In detail, we have a mixture of matter (both collisionless dark particles and baryonic plasma) and relativistic particles (collisionless neutrinos and collisional photons), which does not behave as a simple fluid. Particular problems are caused by the change in the photon component from being a fluid tightly coupled to the baryons by Thomson scattering, to being collisionless after recombination. Accurate results require a solution of the Boltzmann equation to follow the evolution of the full phase-space distribution. This was first computed accurately by Bond & Szalay (1983), and is today routinely available via public-domain codes such as CMBFAST.

Some illustrative results are shown in figure 16. Leaving aside the isocurvature models, all adiabatic cases have $T \to 1$ on large scales – i.e. there is growth at the universal rate (which is such that the amplitude of potential perturbations is constant until the vacuum starts to be important at $z \lesssim 1$). The different shapes of the functions can be understood intuitively in terms of a few special length scales, as follows:

(1) Horizon length at matter-radiation equality. The main bend visible in all transfer functions is due to the Mészáros effect, which arises because the universe is radiation dominated at early times. The relative diminution in fluctuations at high k is the amount of growth missed out on between horizon entry and $z_{\rm eq}$, which would be $\delta \propto D_{\rm H}^2$ in the absence of the Mészáros effect. Perturbations with larger k enter the horizon when $D_{\rm H} \simeq 1/k$; they are then frozen until $z_{\rm eq}$, at which point they can grow again. The missing growth factor is just the square of the change in $D_{\rm H}$ during this period, which is $\propto k^2$. The approximate limits of the CDM transfer function are therefore

$$T_k \simeq \begin{cases} 1 & kD_{\rm H}(z_{\rm eq}) \ll 1 \\ [kD_{\rm H}(z_{\rm eq})]^{-2} & kD_{\rm H}(z_{\rm eq}) \gg 1. \end{cases} \tag{245}$$

This process continues until the universe becomes matter dominated. We therefore expect a characteristic 'break' in the fluctuation spectrum around the comoving horizon length at this time, which we have seen is $D_{\rm H}(z_{\rm eq}) = 16\,(\Omega_m h^2)^{-1}$Mpc. Since distances in cosmology always scale as h^{-1}, this means that $\Omega_m h$ should be observable.

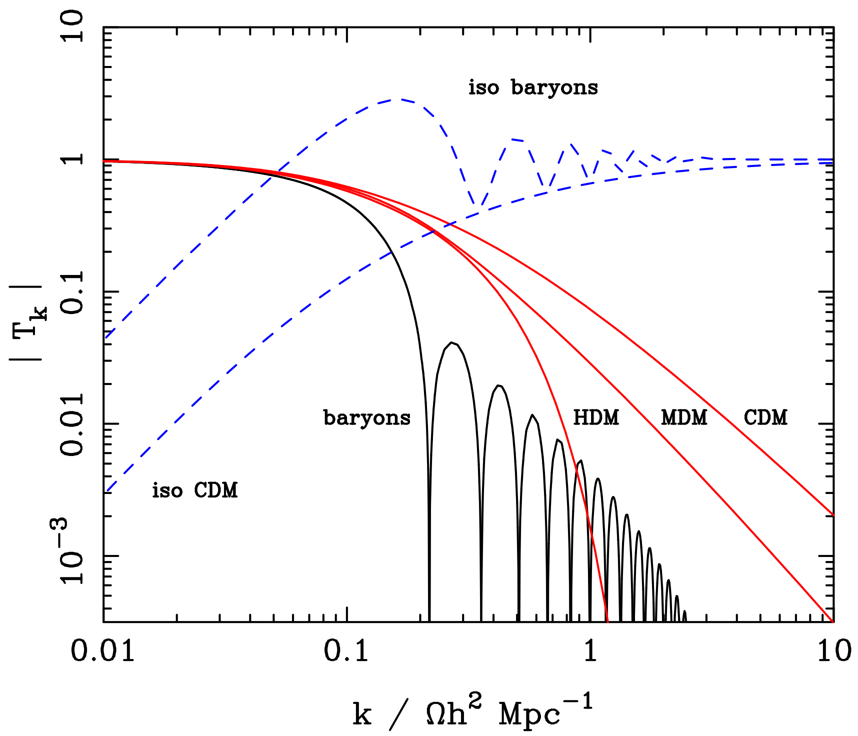

Figure 16. A plot of transfer functions for various adiabatic models, in which $T_k \to 1$ at small k. A number of possible matter contents are illustrated: pure baryons; pure CDM; pure HDM. For dark-matter models, the characteristic wavenumber scales proportional to $\Omega_m h^2$, marking the break scale corresponding to the horizon length at matter-radiation equality. The scaling for baryonic models does not obey this exactly; the plotted case corresponds to $\Omega_m = 1$, $h = 0.5$.

(2) Free-streaming length. This relatively gentle filtering away of the initial fluctuations is all that applies to a universe dominated by Cold Dark Matter, in which random velocities are negligible. A CDM universe thus contains fluctuations in the dark matter on all scales, and structure formation proceeds via hierarchical process in which nonlinear structures grow via mergers. Examples of CDM would be thermal relic WIMPs with masses of order 100 GeV, but a more interesting case arises when thermal relics have lower masses. For collisionless dark matter, perturbations can be erased simply by free streaming: random particle velocities cause blobs to disperse. At early times ($kT > mc^2$), the particles will travel at c, and so any perturbation that has entered the horizon will be damped. This process switches off when the particles become non-relativistic, so that perturbations are erased up to proper lengthscales of $\simeq ct(kT = mc^2)$. This translates to a comoving horizon scale ($2ct/a$ during the radiation era) at $kT = mc^2$ of

$$L_{\rm free-stream} = 112\,(m/{\rm eV})^{-1}\,{\rm Mpc} \tag{246}$$

(in detail, the appropriate figure for neutrinos will be smaller by $(4/11)^{1/3}$ since they have a smaller temperature than the photons). A light neutrino-like relic that decouples while it is relativistic satisfies

$$\Omega_\nu h^2 = m/94.1\,{\rm eV} \tag{247}$$

Thus, the damping scale for HDM (Hot Dark Matter) is of order the bend scale. The existence of galaxies at $z \simeq 6$ tells us that the coherence scale must have been below about 100 kpc, so the DM mass must exceed about 1 keV.

A more interesting (and probably practically relevant) case is when the dark matter is a mixture of hot and cold components. The free-streaming length for the hot component can therefore be very large, but within range of observations. The dispersal of HDM fluctuations reduces the CDM growth rate on all scales below $L_{\rm free-stream}$ – or, relative to small scales, there is an enhancement in large-scale power.

(3) Acoustic horizon length. The horizon at matter-radiation equality also enters in the properties of the baryon component. Since the sound speed is of order c, the largest scales that can undergo a single acoustic oscillation are of order the horizon. The transfer function for a pure baryon universe shows large modulations, reflecting the number of oscillations that have been completed before the universe becomes matter dominated and the pressure support drops. The lack of such large modulations in real data is one of the most generic reasons for believing in collisionless dark matter. Acoustic oscillations persist even when baryons are subdominant, however, and can be detectable as lower-level modulations in the transfer function. We will say more about this later.

(4) Silk damping length. Acoustic oscillations are also damped on small scales, where the process is called Silk damping: the mean free path of photons due to scattering by the plasma is non-zero, and so radiation can diffuse out of a perturbation, convecting the plasma with it. The typical distance of a random walk in terms of the diffusion coefficient D is $x \simeq \sqrt{Dt}$, which gives a damping length of

$$\lambda_{\rm S} \simeq \sqrt{\lambda D_{\rm H}}, \tag{248}$$

the geometric mean of the horizon size and the mean free path. Since $\lambda = 1/(n\sigma_{\rm T}) = 44.3(1+z)^{-3}(\Omega_b h^2)^{-1}$ proper Gpc, we obtain a comoving damping length of

$$\lambda_{\rm S} = 16.3\,(1+z)^{-5/4}(\Omega_b^2 \Omega_m h^6)^{-1/4}\ {\rm Gpc}. \tag{249}$$

This becomes close to the horizon length by the time of last scattering, $1+z \simeq 1100$. The resulting damping effect can be seen in figure 16 at $k \sim 10k_{\rm H}$.