

Angular momentum transport in accretion discs

Pedagogical seminar

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1 Introduction

Accretion discs are ubiquitously found around a variety of astrophysical systems: Young Stellar Objects (YSO), supermassive black holes in AGN, binary systems, stellar mass compact objects in our Galaxy, Saturn’s rings and etc. Also galactic discs can be considered as special cases of accretion discs. Generally they play very important role, for example, accretion discs around compact objects determine emitted radiation from these objects (e.g., X-ray sources) and in the case of YSO they represent sites of planet formation. Accretion disc forms because infalling on a central object matter always has some angular momentum, so it cannot accrete directly unless somehow gets rid of the latter. As a result, the matter settles into a flattened rotating configuration, or a disc. After settling into disc shape, accretion primarily proceeds through such a disc by redistribution of angular momentum so that matter nearer the central object and falling onto its surface gives up angular momentum to outer parts of the disc. During this process disc spreads, because small amount of matter should eventually carry all the angular momentum outwards, while the rest of the mass losing angular momentum falls onto the star (Lynden-Bell & Pringle 1974, Pringle 1981).

Usually discs are subject to various instabilities such as: gravitational, magnetorotational (MRI), purely hydrodynamical and convective instabilities that, in turn, cause turbulence in discs that ensures outward transport of angular momentum. The effect of turbulence on the angular momentum transport can be characterized by so-called *turbulent viscosity* (we will show below that ordinary molecular viscosity of gas is far too small to yield typical timescales of secular evolution of accretion discs). Turbulence plays a dual role here, first it is responsible for outward angular momentum transport necessary for accretion to proceed; instabilities when developed into turbulent regime exert strong torques on the disc able to redistribute angular momentum to large radii. Secondly, it provides a channel of conversion of gravitational energy liberated as mass falls onto the star into thermal energy. The dissipated energy, in turn, contributes to the emitted radiation that can be observed and measured. However, here we do not go into details about an exact origin of turbulent viscosity. Instead, we describe in detail the process of the above mentioned redistribution of angular momentum brought about by the action of such turbulent viscosity by adopting for the latter different phenomenological expressions. This allows us to understand key features of viscous evolution of accretion discs without knowledge of an exact origin of viscosity.

2 Basic equations

In order to get insight into how viscosity (irrespective of its origin) acts to transport angular momentum outwards, following Lynden-Bell & Pringle (1974), Pringle (1981) and Frank et al. (2002), let us start with the simplest model of accretion disc – a razor thin gaseous disc rotating around a central star with considerably larger mass M so that disc self-gravity can be neglected. We adopt cylindrical polar coordinates (r, ϕ, z) with central star at the origin. The disc lies in the $z = 0$ plane. The razor thin disc approximation implies that the disc scale height $H = c_s/\Omega$, where again c_s is the sound speed and Ω is the angular velocity of disc rotation (this expression for H is easily obtained from the vertical hydrostatic equilibrium) is much smaller than the distance from the central star $H/r \ll 1$. In protoplanetary discs, for example, this ratio is typically $H/r \sim 0.05-0.1$ and in AGN discs it is even smaller, $H/r \sim 0.001-0.01$, so the condition for thin disc limit is more or less satisfied. Many studies are limited just to this approximation, because it helps to understand basic physical processes and instabilities, which can be then generalized to thick discs. In this case basic hydrodynamic equations are integrated in the vertical direction to yield two dimensional equations. The disc is then characterized by its surface density Σ , which is a mass per unit surface area of the disc, given by integrating the gas density in the vertical z -direction. The thin disc approximation is equivalent to requiring that the sound speed c_s is much less than the rotation velocity $r\Omega(r)$, or the disc flow is strongly supersonic. From this condition one can readily find that the rotation velocity is given by Kepler law (Pringle 1981, Frank et al. 2002):

$$\Omega(r) = \Omega_K(r) = \left(\frac{GM}{r^3} \right)^{1/2}.$$

In this case a disc is said to be rotationally supported, i.e., gravity force exerted by the central star is balanced by centrifugal force due to rotation. Because of the thin disc approximation, the pressure

gradients turn out to be much smaller than these two forces. The disc being accreting implies that in addition to Keplerian velocity it also possesses radial, or 'drift' velocity v_r directed to the central star. As will be clear below, this radial velocity is much smaller than Keplerian velocity so the ordering $v_r \ll c_s \ll r\Omega_K$ holds and is directly related to the viscosity parameter ν implying that accretion takes place on time scales longer than dynamical time Ω^{-1} . The disc is assumed to be axisymmetric, so that all variables are functions of only radius r and time t (z dependence is absorbed by vertical averaging). The continuity equation for the surface density $\Sigma(r, t)$ is:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0. \quad (1)$$

We also need another equation for angular momentum conservation, which can be similarly derived. The only difference is that we should include now viscous forces proportional to ν . As a result, we obtain:

$$\frac{\partial}{\partial t} (\Sigma r^2 \Omega) + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r r^2 \Omega) = \frac{1}{r} \frac{\partial}{\partial r} (\nu \Sigma r^3 \Omega'), \quad (2)$$

where the prime at Ω denotes derivative with respect to r . The term on the right hand side represents torques exerted by viscous forces on gas. Eqs (1)-(2) can be combined to yield a single equation for surface density evolution:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \right]. \quad (3)$$

In deriving this equation we have made use of the fact that the rotation is stationary and Keplerian. This is the basic equation governing the time evolution of surface density, or mass transport, in a Keplerian disc due to some kind (generally turbulent) of viscosity characterized by ν . In general, equation (3) is a nonlinear diffusion-type equation for Σ , because ν is not necessarily constant and can be a very complex function of local variables (surface density, radius, temperature, ionization fraction etc.) as it typically happens for effective viscosity due to turbulence. So, generally this equation should be addressed numerically. However, if viscosity can be expressed as a power of radius then analytic solutions are feasible (Lynden-Bell & Pringle 1974). This simple case allows us to get the general feeling of angular momentum evolution. In the following subsections we consider different prescriptions for viscosity.

The radial velocity is proportional to viscosity and therefore is small:

$$v_r = -\frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \sim \frac{\nu}{r}. \quad (4)$$

We can also estimate a typical timescale of viscous/secular evolution of a disc

$$t_{visc.} \sim \frac{r}{v_r} \sim \frac{r^2}{\nu}.$$

It is easy to derive the energy dissipated due to viscosity per unit area per unit time at some r , or energy dissipation rate,

$$Q(r) = \frac{1}{2} \nu \Sigma (r \Omega')^2. \quad (5)$$

Obviously, this heat produced during the accretion process ultimately originates from the gravitational energy released as the matter falls into the gravitation well of the central object and contributes to the formation of emitted spectrum that is an important observational characteristic of discs (see section 4). The above analysis shows that viscosity is very important for disc evolution. It is one of the main agents responsible for outward transport of angular momentum. However, the major unresolved problem here is the physical origin of ν viscosity. It was soon realized that standard molecular viscosity is unable to provide appropriate timescales for disc evolution; it is far too small and therefore viscous time scale in this case too big compared with observed disc lifetimes. To demonstrate the case, let us do some estimates for protoplanetary discs around YSO. Indeed, standard molecular viscosity is given by $\nu \sim \lambda c_s$ and the ratio of viscous and dynamical times by $t_{visc}/t_{dyn} \sim r^2/(H\lambda) \sim Re$, where $Re \equiv \Omega r^2/\nu$ is the Reynolds number of disc flow, the mean free path for a typical protoplanetary disc is $\lambda \sim 1/n\sigma = 2.5$ cm (for the number density $n \sim 4 \cdot 10^{14}$ cm $^{-3}$ of molecules having cross-section $\sigma \sim 10^{-15}$ cm 2) and $H/r \sim 0.05$. If we

now take $r \sim 1$ AU, we obtain $t_{visc}/t_{dyn} = Re \sim 10^{14}$. Since the dynamical time scale for protoplanetary discs is of the order of a few years, the t_{vis} turns out to be of the order of age of Universe. Evidently, molecular viscosity is so small that can not provide an adequate evolutionary timescale comparable to the disc lifetime $10^6 - 10^7$ yr. This means that there should exist some other mechanisms capable of producing orders of magnitude larger, or anomalous, viscosity and angular momentum transport that will be consistent with disc lifetime measurements.

The very large Reynolds number $Re \sim 10^{14}$ for disc flows may offer a clue to resolve this problem. It is well known from laboratory experiments that if the Reynolds number is gradually increased, it can reach a critical value typically of the order of $10^3 - 10^4$ after which the flow becomes turbulent: velocity undergoes large and chaotic variations on arbitrary short time and length scales. Thus, given such a huge value for disc Reynolds number compared with its threshold values in laboratory experiments, it is natural to assume that disc flow is strongly turbulent as well. In this case the turbulent flow will be characterized by the size λ_t and turnover velocity v_t of largest eddies. This turbulent flow is superimposed on the main background Keplerian rotation just as small chaotic motions of molecules are superimposed on the mean laminar flow. In analogy with this picture we can define the turbulent viscosity as $\nu_t = \lambda_t v_t$. Because the size of turbulent eddies λ_t is many orders of magnitude larger than mean free path, we expect the turbulent viscosity to be considerably larger too than the molecular viscosity so that it can provide anomalous transport of angular momentum. However, since generally turbulence is a very complex phenomenon, we do not know how to exactly determine the sizes and velocities of eddies without proper knowledge of what are the underlying physical mechanisms causing the transition to turbulence in discs. Although, so far we have assumed its hydrodynamic origin, as detailed investigations have shown over the last two decades, turbulence and, therefore turbulent viscosity mostly occurs due to magnetorotational and gravitational instabilities and possibly due to convective instability too. The reason for not believing in its purely hydrodynamical origin is that it has been shown in many studies that Keplerian differential rotation of a gas in the absence of magnetic fields and self-gravity is stable both linearly and nonlinearly that, in turn, precludes the onset of turbulence. Nevertheless, whatever the origin of turbulent viscosity, we can put constraints on these two quantities. First, the typical size of the largest turbulent eddies can not exceed disc thickness H , otherwise turbulence would not be isotropic and transport local, i.e., the characterization of turbulent transport in terms of viscosity coefficient would not be possible; so $\lambda_t < H$. Secondly, it is unlikely that turbulence in discs is supersonic, because in this case strong shocks would appear and damp turbulence to subsonic values; so $v_t < c_s$. Based on these two basic arguments, Shakura & Sunyaev (1973) suggested the following parameterization of turbulent viscosity:

$$\nu = \alpha c_s H,$$

where α is some constant generally less than unity, because of the above discussed constraints on the size and velocity of turbulent eddies. This formula is so-called α -prescription for turbulent viscosity. Observations indicate that $\alpha \sim 0.01$ for protoplanetary discs, which yields disc evolutionary time scale exactly of the order of disc lifetime. In fact, α contains in itself all the uncertainties regarding the onset mechanism and properties of turbulence in discs. Nonetheless, the α -prescription has provided a useful parametrization of our ignorance of an exact nature of turbulence and has encouraged a semi-empirical approach to the viscosity problem, which seeks to estimate the magnitude of α by a comparison of theory and observations. However, below, as noted above, we consider only general consequences of viscous evolution adopting different phenomenological expressions without knowing its exact origin.

3 Viscous evolution

3.1. Constant viscosity

In order to get first insight in the effects of viscosity on angular momentum transport and secular evolution of a disc, let us first consider a highly simplified case of spatially uniform $\nu = const$. Here we follow Frank et al. (2002) in derivation of equations. Then Eq. (3) can be rewritten as

$$\frac{\partial}{\partial t}(r^{1/2}\Sigma) = \frac{3\nu}{r} \left(r^{1/2} \frac{\partial}{\partial r} \right)^2 (r^{1/2}\Sigma),$$

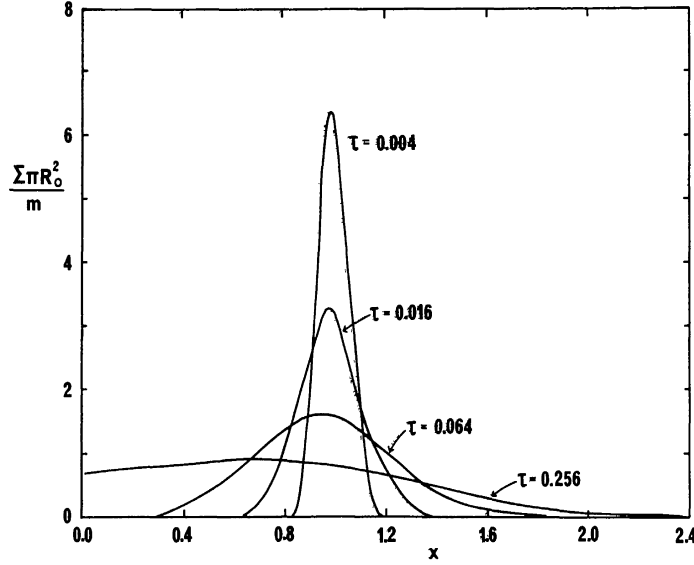


Figure 1: The viscous evolution of a ring of matter of mass m placed initially in a Keplerian orbit at $r = r_0$. As evident, the ring gradually spreads out under the action of viscous torques. The surface density is shown as a function of normalized radius $x = r/r_0$ at four different moments $\tau = 0.004, \tau = 0.016, \tau = 0.064, \tau = 0.256$, where $\tau = 12\nu t/r_0^2$ (after Pringle 1981).

thus setting $s = 2r^{1/2}$, this equation becomes

$$\frac{\partial}{\partial t}(r^{1/2}\Sigma) = \frac{12\nu}{s^2} \frac{\partial^2}{\partial s^2}(r^{1/2}\Sigma),$$

Hence we can separate variables $r^{1/2}\Sigma = T(t)S(s)$, finding

$$\frac{T'}{T} = \frac{12\nu}{s^2} \frac{S''}{S} = const = -\lambda^2.$$

The separated functions T and S are therefore exponentials and Bessel functions respectively.

It is interesting to find the Green function, which is by definition, the solution for $\Sigma(r, t)$ taking as the initial matter distribution a ring of mass m at $r = r_0$:

$$\Sigma(r, t = 0) = \frac{m}{2\pi r_0} \delta(r - r_0),$$

where $\delta(r - r_0)$ is the Dirac delta function. Standard methods give

$$\Sigma(x, \tau) = \frac{m}{\pi r_0^2} \tau^{-1} x^{-1/4} \exp\left(-\frac{1+x^2}{\tau}\right) I_{1/4}(2x/\tau) \quad (6)$$

where $I_{1/4}(z)$ is a modified Bessel function and we have used the dimensionless radius and time variables $x = r/r_0$, $\tau = 12\nu t/r_0^2$. Fig. 1 shows $\Sigma(x, \tau)$ as a function of x for various values of τ . We can see from this figure that viscosity has the effect of spreading the original ring in radius on a typical time scale

$$t_{visc} \sim r^2/\nu$$

as estimated above from a general formula (4). This obtained by setting $(1+x^2)\tau^{-1} \sim x^2\tau^{-1} \sim 1$ in the argument of the exponential in Eq. (6). Now that Σ is known, we can easily find radial velocity determining mass inflow into the center. From Eq. (4) we have for $\nu = const$:

$$v_r = -3\nu \frac{\partial}{\partial r} \ln(r^{1/2}\Sigma) = -\frac{3\nu}{r_0} \frac{\partial}{\partial x} \ln(x^{1/2}\Sigma) =$$

$$= -\frac{3\nu}{r_0} \frac{\partial}{\partial x} \left(\frac{1}{4} \ln x - \frac{1+x^2}{\tau} + \ln I_{1/4} \left(\frac{2x}{\tau} \right) \right).$$

The asymptotic behaviour, $I_{1/4}(z) \propto z^{-1/2} e^z$ at $z \gg 1$ and $I_{1/4}(z) \propto z^{1/4}$ at $z \ll 1$, now shows that

$$v_r \sim \frac{3\nu}{r_0} \left(\frac{1}{4x} + \frac{2x}{\tau} - \frac{2}{\tau} \right) > 0, \quad \text{at } 2x \gg \tau$$

$$v_r \sim -\frac{3\nu}{r_0} \left(\frac{1}{2x} - \frac{2x}{\tau} \right) < 0, \quad \text{at } 2x \ll \tau.$$

Hence, the outer parts of the matter distribution ($2x \gg \tau$) move outwards, taking away angular momentum of the inner parts, which move inwards towards the accreting star. Moreover, the radius at which v_r changes its sign moves outwards itself, for if we choose some point at which x initially much greater than τ , after sufficient time this same point will have x much less than τ . Thus parts of the matter distribution which are at radii $r > r_0$ just after the initial release of the ring ($t \sim 0$) at first move to larger radii, but later begin to lose angular momentum to parts of the disc at still larger radii and thus drift inwards. At very long times ($\tau \gg 1$) after the initial release, almost all of the original m has accreted on to the central star ($r \sim 0$), while all of the original angular momentum has been carried to very large radii by a very small fraction of the mass. We will confirm that this general trend of disc viscous evolution holds even for radially varying viscosity coefficient.

3.2. Radially varying viscosity

Now consider more general case where ν is given as a power law of radius,

$$\nu = \nu_0 \left(\frac{r}{r_0} \right)^\gamma \equiv c_0 r^\gamma.$$

Such a dependence is expected from the fact that typically in discs sound speed has a power law dependence itself and, hence scale height also follows this trend (angular velocity of disc rotation is Keplerian $\equiv r^{-3/2}$, while the α is approximately constant with radius). For the convenience of further analysis, following Lynden-bell & Pringle (1974) let us change independent variables: $h = r^{1/2}$ and $g = \nu \Sigma r^{1/2}$. After this Eq. (3) is rewritten as

$$\frac{\partial^2 g}{\partial h^2} = \frac{4h^{2(1-\gamma)}}{3c_0} \frac{\partial g}{\partial t}.$$

If we now assume that time variation is given by $g \propto e^{-st}$, we have

$$\frac{\partial^2 g}{\partial h^2} + \frac{4sh^{2(1-\gamma)}}{3c_0} g = 0.$$

If we now set $x = h^{2-\gamma}$ and $g_1 = x^{1/2(\gamma-2)} g$, we again obtain Bessel's equation

$$\frac{d^2 g_1}{dx^2} + \frac{1}{x} \frac{dg_1}{dx} + \left(\frac{4s}{3c_0} - \frac{1}{4x^2} \right) \frac{g_1}{(\gamma-2)^2} = 0$$

The general solution for this equation is given in terms of Bessel functions for each s supplemented with boundary condition that g vanishes at the origin $r = 0$ (or more precisely at the edge of the boundary layer between disc and star surface). On the other hand, time-dependent solution is a linear combination of Bessel functions with different s . However, from this general solution its hard to extract information on the key properties of angular momentum transport. In order to clearly understand the latter we restrict ourselves to the time evolution of a special solution resulting from the following initial distribution for g :

$$g(h, 0) = Ch \cdot \exp(-ah^{2(2-\gamma)}) = Cx^{1/(2-\gamma)} \exp(-ax^2),$$

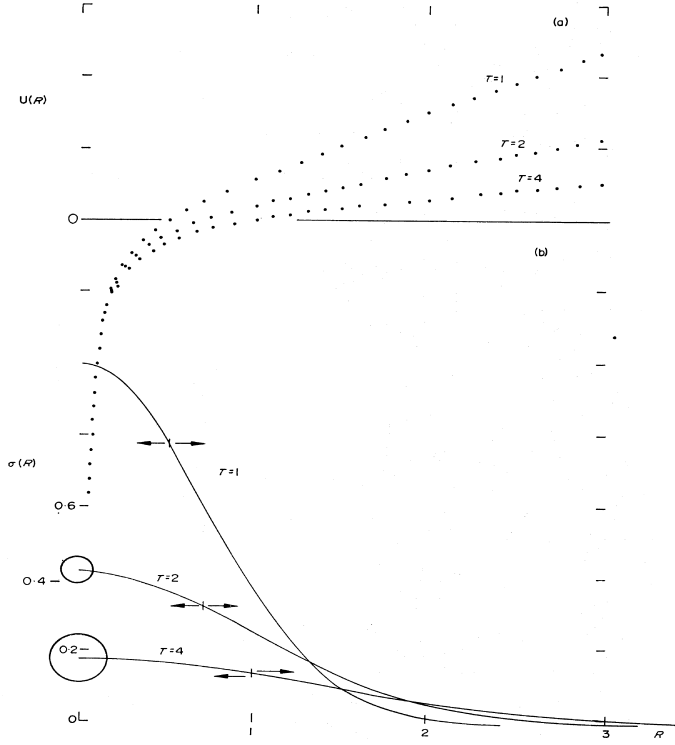


Figure 2: Radial velocity (upper plot) and surface density (lower plot) as a function of radius (here these variables are denoted as u and σ instead of v_r and Σ respectively). The distributions are plotted for three different moments and spheres at $r = 0$ have volumes corresponding to the mass that has fallen from the disc onto the central star (after Lynden-Bell & Pringle 1974).

where C and a are some positive constants. Since g should be vanishing at large distances, we require $\gamma < 2$. It is easy to show that this is equivalent to the initial surface density distribution of the form

$$\Sigma(r, 0) = \frac{C}{c_0 r^\gamma} \exp(-ar^{2-\gamma})$$

After some algebra we arrive at the following analytic solution

$$g(h, t) = CT^{-\frac{(2\gamma-5)}{2(\gamma-2)}} x^{1/(2-\gamma)} \exp(-ax^2/T),$$

where $T = 1 + 3ac_0t(\gamma - 2)^2$ is a dimensionless time. In terms of the physical variables this solution is

$$g = CT^{-\frac{2\gamma-5}{2(\gamma-2)}} h \cdot \exp(-ah^{2(2-\gamma)}/T)$$

$$\Sigma = \frac{g}{h\nu} = \frac{g}{c_0 h^{2\gamma+1}} \quad (7)$$

$$v_r = -\frac{3}{2\Sigma h^2} \frac{dg}{dh} = -\frac{3c_0 h^{2\gamma-1}}{2g} \frac{dg}{dh} = -\frac{3c_0 h^{2(\gamma-1)}}{2} \left(1 - \frac{2(2-\gamma)ah^{2(2-\gamma)}}{T} \right).$$

We see from this formula that at a given moment the radial velocity is directed outwards for $h > (T/2a(2-\gamma))^{1/2(2-\gamma)}$ and inwards inside that radius. This point of velocity reversal moves out as T increases and overtakes more and more material. As a result, at asymptotically large times almost all material moves inwards and only a negligible amount moves outwards carrying all the angular momentum.

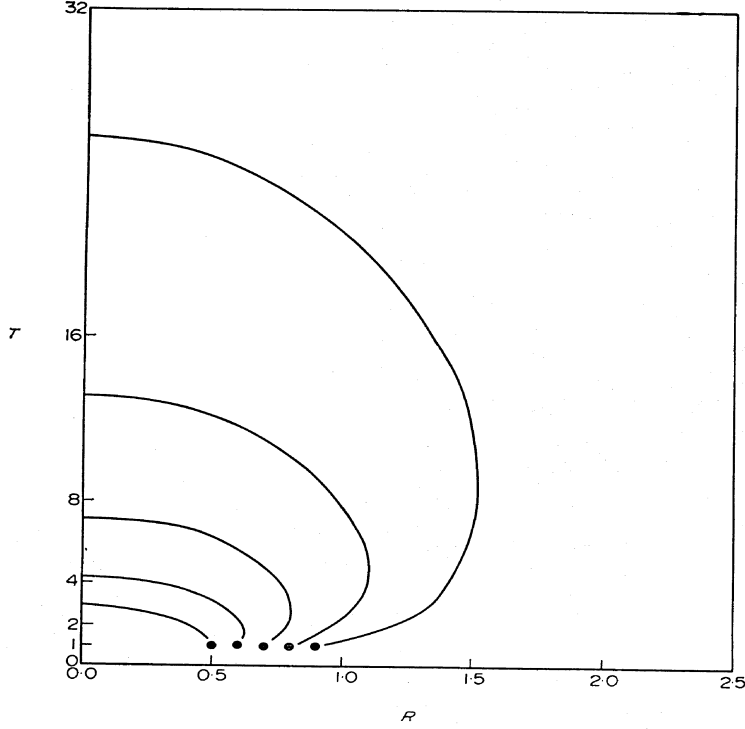


Figure 3: Trajectories of different fluid elements of the disc as functions of time. The fluid elements that start with negative radial velocity move inwards at all times, whereas elements that start with positive radial velocity first move outwards, give up angular momentum to farther elements, then turn and move inwards (after Lynden-Bell & Pringle 1974).

To illustrate this let us calculate the angular momentum distribution in this solution. The total angular momentum within r is given by

$$l = \sqrt{GM} \int_0^r 2\pi r' \Sigma h(r') dr' = \frac{4\pi\sqrt{GM}}{c_0} \int_0^h gh'^{3-2\gamma} dh',$$

or if we introduce the variable $y = ah^{2(2-\gamma)}/T$,

$$l = \frac{4C\pi\sqrt{GM}}{c_0 a^{(2\gamma-3)/(2(\gamma-2))}} \int_0^{y(T,h)} y'^{\frac{1}{2(2-\gamma)}} \exp(-y') dy'.$$

This is a similarity solution for the distribution of l with y remains invariant. Notice that points with constant y , and hence constant l , move outwards with time. This also implies that total angular momentum within a fixed radius r decreases with time simply because it flows outward. We can also see that the angular momentum contained between two rings with constant y_1 and y_2 remain unchanged. Let us see now what happens to the mass between these two rings. It easy to see that this mass is given by

$$M_{12} = \frac{4\pi C}{2(2-\gamma)ac_0} T^{-1/2(2-\gamma)} \int_{y_1}^{y_2} \exp(-y) dy.$$

We see that total mass in such a annulus of constant angular momentum decreases with time as $T^{-1/2(2-\gamma)}$. At the same time the annulus itself travels outward so at asymptotically large times mass content within it is negligible but it still carries the same angular momentum. Analogously it can be shown that the total mass of the disc decreases as $T^{-1/2(2-\gamma)}$. The decrease in mass is accounted by the mass flux into

the origin. We have seen that the radial velocity also changes sign and more and more material is swept inwards. Thus, the real matter distribution is a growing central point mass surrounded by a density distribution which grows in size but decreases in mass. A particular case is illustrated in Fig. 2.

As we have seen, the point of radial velocity reversal moves outwards overtaking more and more material. Thus material that starts inside will move towards the origin whereas material that starts outside will begin by moving outwards only to be overtaken (if it is not the extreme edge). Once overtaken it moves inwards and ends up into the center $r = 0$. The trajectories of fluid elements illustrate this quite clearly. If we follow the trajectory of a single particle we have

$$\frac{dh}{dt} = \frac{dh}{dr} v_r = -\frac{3c_0 h^{2\gamma-3}}{4} \left(1 - \frac{2(2-\gamma)ah^{2(2-\gamma)}}{T} \right),$$

or

$$2a(2-\gamma) \frac{dh^{2(2-\gamma)}}{dT} = - \left(1 - \frac{2(2-\gamma)ah^{2(2-\gamma)}}{T} \right)$$

which integrates to give

$$h^{2(2-\gamma)} = \frac{T}{a} \left(ah_0^{2(2-\gamma)} - \frac{\ln T}{2(2-\gamma)} \right).$$

This gives the paths traced out by the particles as a function of time. So, those particles that start with positive radial velocity move first outwards and then turn around when

$$\ln T = 2a(2-\gamma)h_0^{2(2-\gamma)} - 1.$$

In Fig.3 we show some trajectories of fluid particles describing this motion. By that we quantitatively confirm the behaviour described only for asymptotically large times at the end of the previous subsection. In conclusion, general secular evolution of a viscous disc proceeds as follows: most of the mass moves inwards losing energy and angular momentum as it does so, but a tail of matter moves outwards to larger radii in order to take up the angular momentum. For the considered case the specific angular momentum h for matter in a circular orbit at radius r is given by $h = \Omega r^2 \propto r^{1/2}$ and so tends to infinity at large radius. Thus, eventually all of the matter initially in the disc ends up at the origin and all of the angular momentum is carried to infinite radius by negligible (zero in the limit) mass.

4 Steady disc and emitted spectrum

We have seen in previous section that changes in the radial structure in a thin disc occur on timescales $t_{visc} \sim r^2/\nu$. In many systems external conditions (e.g., mass transfer rate) change on timescales rather longer than t_{visc} . In this case, a stable disc will settle to a steady-state structure, which can be studied by setting $\partial/\partial t = 0$ in conservation equations (1-3). From Eq. (1) we get

$$r\Sigma v_r = const.$$

Clearly, as an integral of the mass conservation equation, this represents the constant inflow of mass through each point of the disc. Since $v_r < 0$ we write

$$\dot{M} = 2\pi r \Sigma (-v_r), \tag{8}$$

where $\dot{M}(gs^{-1})$ is the accretion rate. From the angular momentum equation (2) we get:

$$-\nu \Sigma \Omega' = \Sigma (-v_r) \Omega + \frac{C}{2\pi r^3}. \tag{9}$$

The constant C is related to the rate at which angular momentum flows into the star, or equivalently, the couple exerted by the star on the inner edge of the disc. For example, let us suppose that our disc extends all the way down to the surface $r = r_*$ of the central star. In a realistic situation, the star must rotate more slowly than break-up speed at its equator, i.e., with angular velocity

$$\Omega_* < \Omega_K(r_*).$$

In this case the angular velocity of disc material remains Keplerian and thus increases inwards, until it begins to decrease to the value Ω_* in a 'boundary layer' of very small radial extent, $b \ll r_0$. Hence, there exists a radius $r = r_* + b$ at which $\Omega' = 0$. Because $b \ll r_*$, Ω is very close to its Keplerian value at the point where $\Omega' = 0$, i.e.,

$$\Omega(r_* + b) = \left(\frac{GM}{r_*^3} \right)^{1/2} (1 + O(b/r_*)). \quad (10)$$

If b is comparable to r_* than thin disc approximation breaks down at $r = r_* + b$; this is no longer the boundary layer, but some kind of thick disc. At $r = r_* + b$, Eq.(9) becomes

$$C = 2\pi r_*^3 \Sigma v_r \Omega(r_* + b),$$

which using Eq. (8) implies

$$C = -\dot{M}(GMr)^{1/2}$$

to terms of order b/r_* . Substituting this into Eq. (9) and setting $\Omega = \Omega_K$, we find

$$\nu \Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{r_*}{r} \right)^{1/2} \right]. \quad (11)$$

this last formula for a steady-state disc with a slowly rotating star has an important and elegant result. If we now use Eq. (5) from Eq. (11) we obtain

$$Q(r) = \frac{3GM\dot{M}}{8\pi r^3} \left(1 - \left(\frac{r_*}{r} \right)^{1/2} \right). \quad (12)$$

Hence, the energy flux through the faces of a steady thin disc is independent of viscosity. This is an important result: as mentioned, dissipated viscous energy $Q(r)$ is a quantity of prime observational significance, and Eq. (12) shows that its dependance on \dot{M}, r , etc. is known, even though we are at present fairly ignorant as to the precise physical nature of the viscosity ν . The result relies on the implicit assumption that ν can adjust itself to give the required \dot{M} . The independence of $Q(r)$ from ν has come about because we were able to use conservation laws to eliminate ν ; clearly the other disc properties (e.g. Σ, v_r , etc.) do depend on ν .

Let us now use Eqs. (5),(12) to find the luminosity produced by the disc between radii r_1 and r_2 . This is

$$L(r_1, r_2) = 2 \int_{r_1}^{r_2} Q(r) 2\pi r dr,$$

the factor 2 coming from the two sides of the disc. From Eq. (12) we get

$$L(r_1, r_2) = \frac{3GM\dot{M}}{2} \int_{r_1}^{r_2} \left(1 - \left(\frac{r_*}{r} \right)^{1/2} \right) \frac{dr}{r^2}$$

which can be evaluated by putting $y = r_*/r$, with the result

$$L(r_1, r_2) = \frac{3GM\dot{M}}{2} \left[\frac{1}{r_1} \left(1 - \frac{2}{3} \left(\frac{r_*}{r_1} \right)^{1/2} \right) - \frac{1}{r_2} \left(1 - \frac{2}{3} \left(\frac{r_*}{r_2} \right)^{1/2} \right) \right]$$

Letting $r_1 = r_*$ and $r_2 \rightarrow \infty$, we obtain the luminosity of the whole disc

$$L_{disc} = \frac{GM\dot{M}}{2r_*} = \frac{1}{2} L_{acc}$$

As we see the total disc luminosity is half the energy available from the accretion process, $GM\dot{M}/r_*$. The discrepancy arises because the matter just outside the boundary layer still retains one half of the potential energy it has lost as kinetic energy. Thus, the rest of the accretion luminosity is emitted in the boundary layer, which we do not treat here in detail (in fact, we ignore here boundary layer by working in the approximation $b = 0$). Thus, the emission from the boundary layer is as important as the emission from the disc itself.

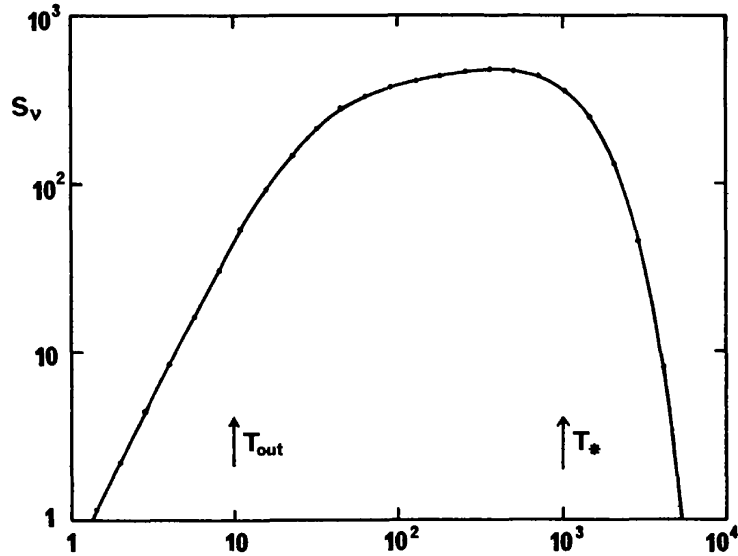


Figure 4: The integrated spectrum of a steady accretion disc that radiates a local blackbody spectrum at each point. The units are arbitrary but we show frequencies corresponding to T_{out} , the temperature of outermost disc radius and to T_* the characteristic temperature of the inner disc (after Pringle 1981). The spreading of the initial ring is clearly seen.

The emitted spectrum

As mentioned, dissipated energy goes into heat that, in turn, is converted into radiation emitted from the disc surface. Thus, in order to find the total spectrum of radiation emitted from the disc we must first determine the spectrum emitted locally at each point in a disc and then integrate to over the whole disc surface. For a proper local determination of a spectrum we should solve radiative transfer equation in the vertical direction at each radius. If the disc is optically thick in the sense that each element radiates as a blackbody with a temperature $T(r)$ then the problem simplifies enormously. Now the corresponding temperature distribution is found by equating dissipation rate $Q(r)$ per unit area to the energy flux for blackbody:

$$\sigma T^4(r) = Q(r).$$

Using Eq. (12) we get

$$T(r) = \left[\frac{3GM\dot{M}}{8\pi r^3} \left(1 - \left(\frac{r_*}{r} \right)^{1/2} \right) \right]^{1/4}.$$

Far from the star's surface, $r \gg r_*$, this formula simplifies to

$$T(r) = T_* \left(\frac{r}{r_*} \right)^{-3/4},$$

where $T_* = (3GM\dot{M}/8\pi\sigma r_*^3)^{1/4}$. Temperature $T(r)$ here plays an analogous role to the effective temperature of a star: very crudely we can approximate the spectrum emitted by each element of area of the disc as (ν is frequency here, not to be confused with viscosity coefficient):

$$I_\nu = B_\nu [T(r)] = \frac{2h\nu^3}{c^2 (e^{h\nu/kT(r)} - 1)}.$$

The approximation of disc being optically thick involved here neglects the effect of the atmosphere of the disc, i.e., that part of the disc material at optical depth $\tau \leq 1$ from infinity in redistributing the

radiation over frequency ν . Hence, this prescription may not represent the detailed spectrum of the disc very well in particular frequency domains, especially those where the atmospheric opacity is a rapidly varying function of frequency. For an observer at a distance D whose line of sight makes an angle i to the normal to the disc plane, the flux at frequency ν from the disc is

$$S_\nu = \frac{2\pi \cos i}{D^2} \int_{r_*}^{r_{out}} I_\nu r dr,$$

where r_{out} is the outer radius of the disc, since a ring between radii r and $r + dr$ subtends solid angle $2\pi r dr \cos i / D^2$. With the blackbody assumption, we get

$$S_\nu = \frac{4\pi h \cos i \nu^3}{c^2 D^2} \int_{r_*}^{r_{out}} \frac{r dr}{e^{h\nu/kT(r)} - 1}. \quad (13)$$

An important feature of this result is that S_ν is independent of the disc viscosity. This is a consequence of both the steady and blackbody assumptions. Since both of these are likely to be at least roughly valid for some systems, we should expect the spectrum specified by Eq. (13) to give at least crude representation of the observed spectrum for some systems, in the same way that stellar spectra are to a crude approximation blackbody: any serious discrepancy cannot be due to ignorance of viscosity.

The resulting S_ν spectrum is shown in Fig. 4. The shape of this spectrum is easy to deduce from Eq. (13). First for frequencies $\nu \ll kT(r_{out})/h$ the Planck function B_ν takes the Rayleigh-Jeans form $2kT\nu^2/c^2$; hence Eq. (13) gives $S_\nu \propto \nu^2$. For $\nu \gg kT_*/h$ Planck function assumes the Wien form $2h\nu^3 c^{-2} e^{-h\nu/kT}$: the integral is dominated by the hottest parts of the disc ($T \sim T_*$) and the integrated spectrum is exponential. For intermediate frequencies ν such that $kT(r_{out})/h \ll \nu \ll kT_*/h$ we let $x = h\nu/kT(r) \simeq (h\nu/kT_*)(r/r_*)^{3/4}$. Then Eq. (13) becomes approximately

$$S_\nu \propto \nu^{1/3} \int_0^\infty \frac{x^{5/3} dx}{e^x - 1} \propto \nu^{1/3}$$

since the upper limit in the integral is $h\nu/kT(r_{out}) \gg 1$ and the lower limit is $h\nu/kT_* \ll 1$. Thus the integrated spectrum S_ν is a stretched out blackbody; the 'flat' part $S_\nu \propto \nu^{1/3}$ is sometimes considered a characteristic disc spectrum. However, unless $T_{out} = T(r_{out})$ is appreciably smaller than T_* this part of the curve may be quite short and the spectrum is not very different from a blackbody.

5 Summary

Here we described the basic effect of viscosity on the secular evolution of accretion discs. Usually this viscosity originates from some kind of turbulence (due to gravitational, magnetorotational, purely hydrodynamical, convection instabilities) in a disc and cannot be ordinary molecular viscosity because the latter is extremely small to give appropriate evolutionary time scales. We demonstrated that irrespective of the origin of such a turbulent viscosity the main evolutionary picture remains unchanged and consists in the following: most of the disc's mass moves inwards losing energy and angular momentum as it does so, but a tail of matter moves outwards to larger and larger radii in order to take up the angular momentum. We also calculated the trajectories of fluid particles clearly showing this behaviour. Based on the steady disc model and blackbody assumption, we calculated the typical spectrum that comes from accretion discs. Of course, this is a highly idealized case and gives only very crude approximation for the emitted spectrum, because we it does not includes modifications due to optically thin parts of the disc. Nevertheless, it a good starting point towards more complicated models of accretion discs.

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