Mapping the Dark Universe with Weak Gravitational Lensing

Rachel N McInnes*

Institute for Astronomy, Royal Observatory of Edinburgh, Blackford Hill, Edinburgh, EH9 3HJ

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*rnm@roe.ac.uk

Figure 1: The bullet cluster. The mass map (dark matter) is shown in blue, and the xray is shown in purple, both on top of the optical image. The dark matter map has been made using gravitational lensing mass reconstruction - a unique way to probe dark matter. You can see that the xray gas has interacted when the two clusters have passed through each other, while the majority of the mass of the 2 clusters has not interacted, and have passed without a collision. This is evidence for collisionless dark matter.

1 Introduction to Lensing Theory

Tidal gravitational fields cause differential deflection of light rays. This means that the size and shape of their projection on the sky are changed. In some cases the deflection can be so strong that luminous *arcs* or *Einstein rings* are observed. And in extreme cases light rays passing close to a sufficiently massive body can be bent so strongly that multiple rays can reach the observer. This means that multiple images of the same object are observed, each in the direction that a ray has arrived. (See Kochanek (2004) for more detail)

Einstein rings, arcs, and multiple images are all effects classed as *Strong* Gravitational Lensing. Weaker gravitational effects occur more often, however. Although these weak distortions are small and can hardly be noticed in an individual image, the net distortion averaged over an area of sky can be detected. These distortion effects due to *Weak* Lensing can be used to give us statistical properties of the matter distribution between us and the lensed source.

In a third Lensing régime —Micro-Lensing — magnification of a distant source can occur. Micro-Lensing can be thought of as a version of Strong Lensing where the image separation is too small to be resolved. Multiple images are formed but their separations are below the limiting resolution of our observations and this causes the source to appear magnified. Micro-lensing occurs for sufficiently small lens masses (such as a star), and for sufficiently distant sources and lenses. This magnification effect is sometimes referred to as a “a cosmic telescope”. This is because it lets us observe objects which would otherwise be too faint or distant to observe. (See Wambsganss (2006) for further detail)

Whether Strong, Weak or Micro-Lensing effects are seen depends on the the size of the *lensing potential*, which we will meet shortly.

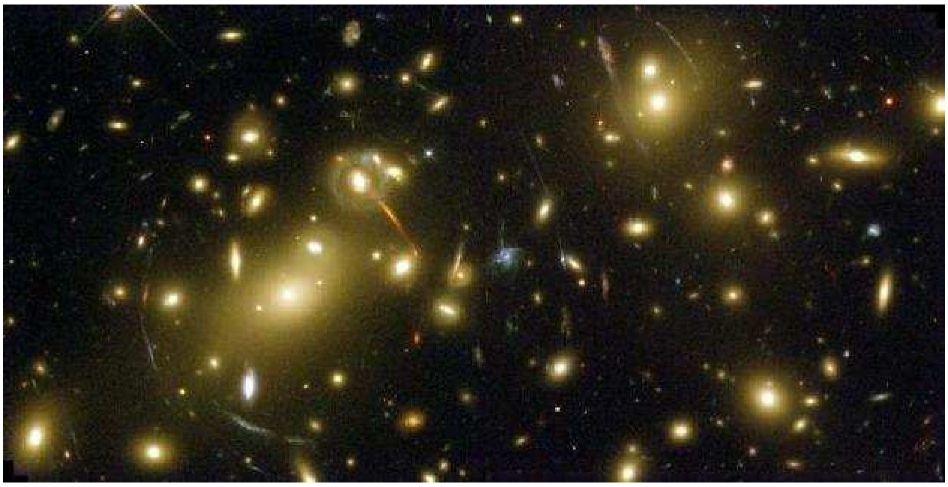

Figure 2: Arcs caused by Gravitational Lensing (Bartelmann, 2006).

1.1 Applications of Gravitational Lensing

Gravitational Lensing is a tool which is used by observational cosmologists. There are two main areas which this tool can be applied: 1) probing dark matter and 2) probing the geometry of the universe.

1. Gravitational Lensing depends only on the projected 2D mass distribution of the lens and not on whether the matter is baryonic matter or dark matter. Dark matter can thus be investigated using this method as the lens is independent of luminosity and composition.

2. Gravitational Lensing not only tells us about the lensing object, but about the space that the light ray has travelled through to reach us from its source. Cosmological parameters including the Hubble constant, the cosmological constant and the density parameter of the universe can all be constrained through lensing.

2 Measurements of Shapes and Shear

There are different ways to measure the shape of an object. The method described here is the one set out in Kaiser, Squires & Broadhurst (1995) and Schneider (2006).

In order to define the ellipticity of an object, we must first define the centre of the image, $\overline{\theta}$,

$$\overline{\boldsymbol{\theta}} = \frac{\int d^2\theta \, I(\boldsymbol{\theta}) q_I[I(\boldsymbol{\theta})]\boldsymbol{\theta}}{\int d^2\boldsymbol{\theta} \, I(\boldsymbol{\theta}) q_I[I(\boldsymbol{\theta})]} \tag{1}$$

where $I(\boldsymbol{\theta})$ is the brightness distribution of the image, and $q_I(I)$ is a weight function.

We next define the tensor of *second brightness moments*,

$$Q_{ij} = \frac{\int d^2\theta \, I(\boldsymbol{\theta}) q_I[I(\boldsymbol{\theta})](\theta_i - \overline{\theta}_i)(\theta_j - \overline{\theta}_j)}{\int d^2\boldsymbol{\theta} \, I(\boldsymbol{\theta}) q_I[I(\boldsymbol{\theta})]} \, , \quad i, j \in \{1, 2\} \, . \tag{2}$$

The trace of Q describes the size of the image, and the traceless part gives us the ellipticity information.

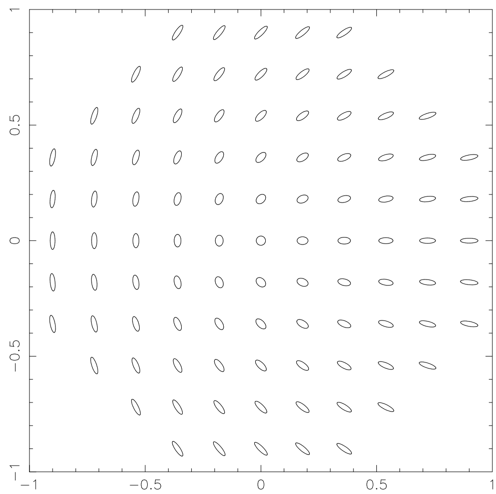

Figure 3: Shape of image ellipses for a circular source, in terms of χ_1 against χ_2. A plot of e_1 against e_2 would look quite similar. (Schneider, 2006)

From Eq. (2), two complex ellipticities can be defined:

$$\chi \equiv \frac{Q_{11} - Q_{22} + 2iQ_{12}}{Q_{11} + Q_{22}} \; ; \quad \epsilon \equiv \frac{Q_{11} - Q_{22} + 2iQ_{12}}{Q_{11} + Q_{22} + 2\left(Q_{11}Q_{22} - Q_{12}^2\right)^{\frac{1}{2}}} \tag{3}$$

Figure 3 shows the shape of image ellipses for a circular source, in terms of their two ellipticity components, χ_1 and χ_2.

To transform between the source and image ellipticities, the following relation can be used (Seitz & Schneider, 1997)

$$\chi^{(s)} = \frac{\chi - 2g + g^2\chi*}{1 + |g|^2 - 2\mathcal{R}e(g\chi*)} \; ; \quad \epsilon^{(s)} = \begin{cases} \frac{\epsilon - g}{1 - g*\epsilon} & \text{if } |g| \leq 1 \; ; \\ \frac{1 - g\epsilon*}{\epsilon* - g*} & \text{if } |g| > 1 \; . \end{cases} \tag{4}$$

where $g = g(z) = \frac{\gamma(z)}{1-\kappa(z)}$ is called the *reduced shear*.

In the weak lensing régime ($\kappa \ll 1, |\gamma| \ll 1$) , the reduced shear $|g| \ll 1$ and Eq. 4 reduces to (where e is ellipticity defined either as χ or ϵ)

$$e' = e^s + 2g \; . \tag{5}$$

If we assume that the sources are randomly orientated then the expectation value of the source ellipticities will vanish and we can write:

$$\langle e^s \rangle = 0 \; . \tag{6}$$

Taking the average of Eq. 5 we see that:

$$\langle e' \rangle = 0 + 2\langle g \rangle \; , \tag{7}$$

and if these sources are in small enough area of sky, we can assume that the light from each of these galaxies experiences the same shear , and so we now can write

$$\gamma \approx g \approx \langle g \rangle = \frac{\langle e' \rangle}{2} \; . \tag{8}$$

Note the factor of 2 here for this definition of ellipticity. [1]

2.1 The Principles of Weak Gravitational Lensing

2.1.1 Light Deflection

In the weak lensing limit we can write the metric as

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) R^2(t) \left[dr^2 + S_k^2(r) d\beta^2\right] , \tag{9}$$

where Φ is the peculiar gravitational potential and the requirement for a weak field is that $|\Phi| \ll c^2$.

For weak fields we use Eq. (9) as our metric and define *conformal time*, in place of the usual time coordinate. Conformal time is defined by

$$d\eta = \frac{cdt}{R(t)} . \tag{10}$$

This allows us to re-write Eq. (9) more simply:

$$ds^2 = R^2(t) \left\{ \left(1 + \frac{2\Phi}{c^2}\right) d\eta^2 - \left(1 - \frac{2\Phi}{c^2}\right) \left[dr^2 + S_k^2(r) \left(d\theta_x^2 + d\theta_y^2\right)\right] \right\} . \tag{11}$$

We can now write the metric tensor for the weakly perturbed flat FLRW metric as

$$R^2(t) \begin{pmatrix} 1 + 2\Phi/c^2 & 0 & 0 & 0 \\ 0 & -(1 - 2\Phi/c^2) & 0 & 0 \\ 0 & 0 & -r^2(1 - 2\Phi/c^2) & 0 \\ 0 & 0 & 0 & -r^2(1 - 2\Phi/c^2) \end{pmatrix} . \tag{12}$$

The most compact way to derive the deflection angle is using the Euler-Lagrange equation. Please see Appendix B for the full derivation. This allows us to write the result:

$$\frac{d^2\boldsymbol{X}}{d\eta^2} = -\frac{2}{c^2} \boldsymbol{\nabla}_\perp \Phi . \tag{13}$$

where $\boldsymbol{\nabla}_\perp = \left(\frac{d}{dX}, \frac{d}{dY}\right)$.

The integral of $-\frac{d^2X}{d\eta^2}$ along the path gives us the *total deflection angle*,

$$\hat{\boldsymbol{\alpha}} = \frac{2}{c^2} \int_A^B \boldsymbol{\nabla}_\perp \Phi d\lambda . \tag{14}$$

In the cases we are interested in, the deflection angle is very small. So we can make a small angle approximation (known as Born's approximation), that the deflection angle is the same if we integrated not along the deflected ray, but along the *unperturbed* ray.

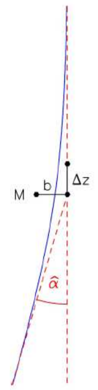

Figure 4: Light deflection by a point mass. The unperturbed ray passes the mass M at impact parameter b and is deflected by angle $\hat{\boldsymbol{\alpha}}$ (Narayan & Bartelmann, 1996).

2.1.2 Deflection Angle of a Point Mass

To illustrate the above situation, we will now consider the deflection angle of a point mass M along the z axis (See Figure 4). The impact parameter, b, is the closest approach distance, and most of the deflection occurs within $\Delta z \sim \pm b$ of the closest approach. The (Newtonian) gravitational potential of the lens is

$$\Phi = -\frac{GM}{r} = -\frac{GM}{(b^2 + z^2)^{\frac{1}{2}}} \ . \tag{15}$$

We now calculate an expression for $\boldsymbol{\nabla}_\perp \Phi$

$$\boldsymbol{\nabla}_\perp \Phi(b, z) = \frac{GM\boldsymbol{b}}{(b^2 + z^2)^{\frac{3}{2}}} \ , \tag{16}$$

and substitute this into Eq. (14) to give us another expression for the deflection angle

$$\hat{\boldsymbol{\alpha}} = \frac{2}{c^2} \int \boldsymbol{\nabla}_\perp \Phi dz = \frac{4GM}{c^2 b} = \frac{2}{b} R_s \ , \tag{17}$$

where R_s is the Schwarzschild radius of the point mass. This allows us to say that the deflection angle is just twice the inverse of the impact parameter in units of the Schwarzschild radius. Note that this deflection angle calculated using General Relativity is exactly twice the value of the angle calculated with Newtonian physics.

2.1.3 The Lens Equation

Figure 5 shows the source plane and the image plane of the gravitational lens system. (Here, and for the following section, please now refer to Narayan & Bartelmann (1996)). $\boldsymbol{\theta}$ is used as coordinates for the image(s) and $\boldsymbol{\beta}$ for the source(s). They are related by Eq. (18)

$$D_s \boldsymbol{\beta} = D_s \boldsymbol{\theta} - D_{ds} \hat{\boldsymbol{\alpha}} \ , \tag{18}$$

[1]Depending on how ellipticity has been defined, you will see in the literature that sometimes $\gamma \approx \langle e' \rangle$ is used.

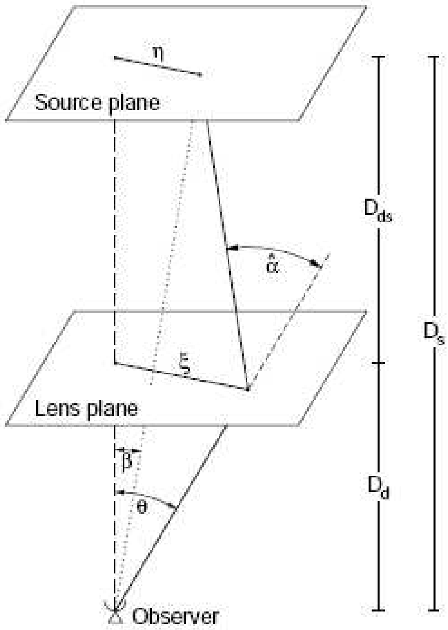

Figure 5: Diagram showing the angles and distances between the observer, lens plane, and source plane (Bartelmann & Schneider, 2001).

where D_d, D_{ds} and D_s are the angular diameter distances between the observer and lens, lens and source, and observer and source respectively.[2]

We now define the *reduced deflection angle*, $\boldsymbol{\alpha}$ and define it to be

$$\boldsymbol{\alpha} \equiv \frac{D_{ds}}{D_s}\hat{\boldsymbol{\alpha}} \,. \tag{19}$$

This allows us to write the simple *lens equation* (or ray-tracing equation):

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha} \,. \tag{20}$$

2.2 Mapping between the Source and the Image Plane

The *local* imaging properties of the lens mapping between the source and the image plane are described by the Jacobian matrix, $\mathcal{A}$. (See Figure 6)

$$\mathcal{A} \equiv \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}} = \mathcal{I} - \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\theta}} \tag{21}$$

$\mathcal{A}$ is also known as the *amplification matrix*. $\mathcal{A}$ is symmetric and we can decompose it into an isotropic and an anisotropic term as follows.

$$\mathcal{A}_{ij} = \begin{bmatrix} 1-\kappa & 0 \\ 0 & 1-\kappa \end{bmatrix} + \begin{bmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{bmatrix} \tag{22}$$

[2]Note that, in general, $D_{ds} \neq D_s - D_d$ because these are angular diameter distances.

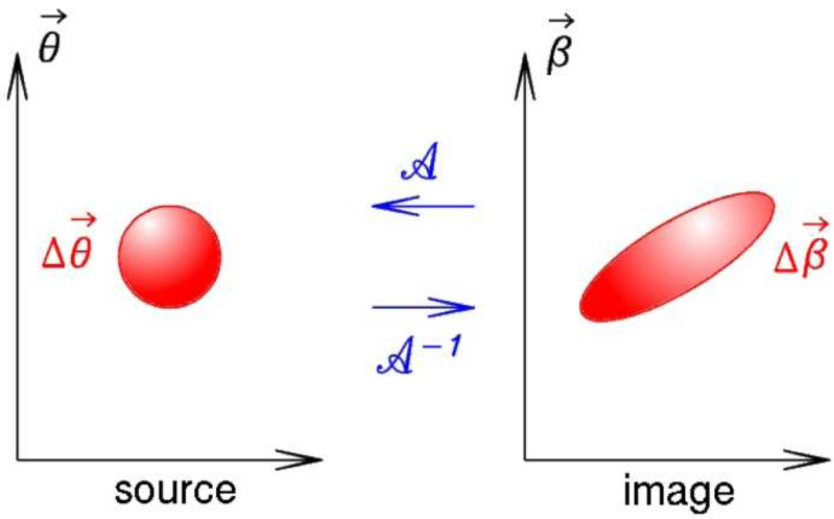

Figure 6: Illustration of how the Jacobian relates the source and the image (Bartelmann, 2006)

The first term is the isotropic expansion expansion term (this is the convergence[3] that we met earlier in the section) and the second term is the *shear* term. An illustration of the effects of each of these is shown in Figure 7.

The amplification matrix can also be written in terms of components of the lensing potential as follows,

$$\mathcal{A}_{ij} = \delta_{ij} - \partial_i \partial_j \varphi \,, \tag{23}$$

where δ_{ij} is the kronecker delta. We can break this down into the diagonal and off-diagonal part as follows:

$$\mathcal{A}_{ij} = \delta_{ij} - \left[\frac{1}{2}\partial^2\varphi\delta_{ij} + \left(\partial_i\partial_j\varphi - \frac{1}{2}\delta_{ij}\partial^2\varphi\right)\right] = \delta_{ij} - \left[\kappa\delta_{ij} + \gamma_{ij}\right] \,. \tag{24}$$

Using the trace of $\mathcal{A}$ and equating Eqs. (22) and (23) lead us to the following:

$$\gamma_1 = \frac{1}{2}\left(\varphi_{11} - \varphi_{22}\right), \tag{25}$$

$$\gamma_2 = \varphi_{12} = \varphi_{21} \,, \tag{26}$$

$$\kappa = \frac{1}{2}\left(\varphi_{11} + \varphi_{22}\right) \,. \tag{27}$$

The effects of gravitational lensing on a circular source are shown in Figure 7.

3 From Shear to Convergence - Making 2D Mass Maps

In this notation, we can express κ and γ as follows

$$\kappa = \frac{1}{2}\partial^2\varphi \,, \tag{28}$$

[3]The term *Convergence* comes from light deflection in empty space. Do remember that, in practice, this can often be an expansion term.

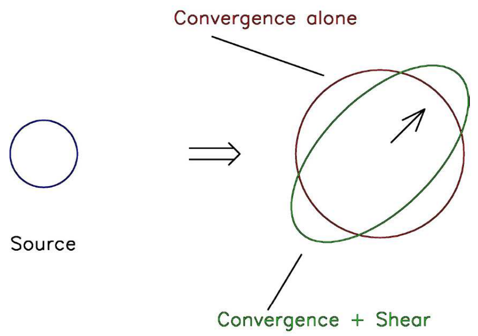

Figure 7: There are two basic types of distortion in gravitational lensing. The convergence which is isotropic, and the shear which is anisotropic (Narayan & Bartelmann, 1996)

Now we can write γ as (Brown et al., 2003)

$$\gamma_{ij} = \begin{bmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{bmatrix} = \partial_i \partial_j \varphi - \frac{1}{2}\partial^2 \varphi \delta_{ij} \, . \tag{29}$$

Using this equation, we perform the operation $\partial_i \partial_j$ to Eq. (29):

$$\partial_i \partial_j \gamma_{ij} = \partial^2 \partial^2 \varphi - \frac{1}{2}\partial^2 \varphi = \frac{1}{2}\partial^2 \partial^2 \varphi = \partial^2 \kappa \tag{30}$$

so we can write the following relation for κ:

$$\kappa = \partial^{-2} \partial_i \partial_j \gamma_{ij} + c \, , \tag{31}$$

where c is a constant of integration.

Using the Fourier Transform Relations below, kappa can be calculated easily using the shear.

Fourier Transform Relations

$$\partial_i = -il_i \, , \tag{32}$$

$$\partial^2 = -|l|^2 \, , \tag{33}$$

$$\partial^{-2} = -\frac{1}{l^2} \, . \tag{34}$$

4 E and B modes

The scalar gravitational potential should only produce a curl-free shear signal. The existence of any significant curl component should be treated as a systematic shear error. We can split the shear (or the convergence) into two modes, one which measures the divergence, and one which measures the curl. We call this *E and B mode decomposition*, with the E mode being the allowed curl-free component and the B mode being the systematic curl component. Figure 8 shows the E modes generated by over-densities (top left) and under-densities (top right). The bottom panels show the two curl modes which are not created by Gravitational Lensing.

You will notice that the B modes are a 45° rotation of the E modes. So to test for systematics (B modes) we can rotate the data by 45° and repeat the same analysis. Anything which is a B mode will now show up as a signal.

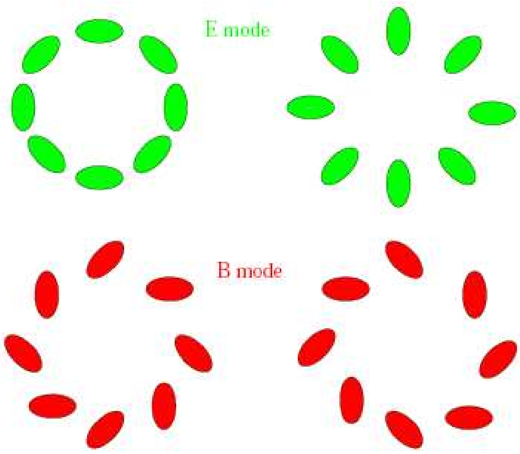

Figure 8: The top panel shows allowed E modes created by Gravitational Lensing. The bottom panel shows B modes which are not allowed by a scalar gravitational potential. (Van Waerbeke & Mellier, 2003).

5 Power Spectra and Correlation

The E and B mode *power spectra* are related to $\kappa_{E,\boldsymbol{l}}$ and $\kappa_{B,\boldsymbol{l}}$ as follows,

$$P^E_{\kappa,\boldsymbol{l}} = \frac{\langle|\kappa_{E,\boldsymbol{l}}|^2\rangle}{N^4 \Delta \ln l} , \tag{35}$$

$$P^B_{\kappa,\boldsymbol{l}} = \frac{\langle|\kappa_{B,\boldsymbol{l}}|^2\rangle}{N^4 \Delta \ln l} . \tag{36}$$

We then define the E and B mode *correlations* as the inverse fast fourier transform of these quantities.

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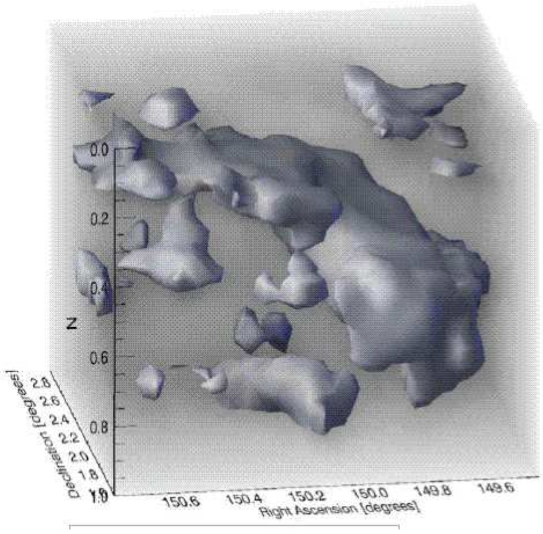

Figure 9: (Massey et al., 2007).

Wambsganss J., eds, Saas-Fee Advanced Course 33: Gravitational Lensing: Strong, Weak and Micro Part 4: Gravitational microlensing. pp 453–540

A Appendix

A.1 Derivation of the deflection angle from the Geodesic Equation

The geodesic equation from General Relativity governs the wordline x^λ ($\lambda = 0, 1, 2, 3$) of a particle. It is written as follows:

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{dp}\frac{dx^\sigma}{dp} = 0\ , \tag{37}$$

where p is an affine parameter. $\Gamma^\mu_{\nu\sigma}$ is the *affine connection*, which can be written in terms of the metric tensor $g_{\mu\nu}$ as

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\rho}\left\{\frac{\partial g_{\nu\rho}}{\partial x^\sigma} + \frac{\partial g_{\rho\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\rho}\right\}\ . \tag{38}$$

Or, equivalently, in alternative notation:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\ . \tag{39}$$

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\rho}\left\{g_{\nu\rho,\sigma} + g_{\rho\sigma,\nu} - g_{\nu\sigma,\rho}\right\}\ . \tag{40}$$

Note that a dot denotes d/dp throughout this section.

For weak fields we use Eq. (9) as our metric and define *conformal time*, in place of the usual time coordinate. Conformal time is defined by

$$d\eta = \frac{cdt}{R(t)}\ . \tag{41}$$

This allows us to re-write Eq. (9) more simply:

$$ds^2 = R^2(t)\left\{\left(1 + \frac{2\Phi}{c^2}\right)d\eta^2 - \left(1 - \frac{2\Phi}{c^2}\right)\left[dr^2 + S_k^2(r)\left(d\theta_x^2 + d\theta_y^2\right)\right]\right\}\ . \tag{42}$$

We can now write the metric tensor for the weakly perturbed flat FLRW metric as

$$R^2(t)\begin{pmatrix} 1 + 2\Phi/c^2 & 0 & 0 & 0 \\ 0 & -(1 - 2\Phi/c^2) & 0 & 0 \\ 0 & 0 & -r^2(1 - 2\Phi/c^2) & 0 \\ 0 & 0 & 0 & -r^2(1 - 2\Phi/c^2) \end{pmatrix}. \tag{43}$$

What we want to find out is how the angles (θ_x, θ_y) of the ray change, as the photon travels along its path, when the varying gravitational potential is present. The path of the unperturbed, radial ray is set by $0 = ds^2 \simeq d\eta^2 - dr^2$, therefore for the radial incoming ray

$$\frac{dr}{d\eta} = -1\ . \tag{44}$$

$g^{\mu\nu}$ is defined such that $g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu$ and so we now compute the affine connections.

We want to find $d^2\eta/dp^2$ so we set $\mu = 0$ in Eq. (39) so we write:

$$\ddot{x}^0 + \Gamma^0_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\ , \tag{45}$$

non-zero affine connections have $\mu = \rho = 0$:

$$\Gamma^0_{\nu\sigma} = \frac{1}{2} g^{0\,0} \left\{ g_{\nu\;0,\sigma} + g_{0\;\sigma,\nu} - g_{\nu\sigma,0} \right\} \; . \tag{46}$$

We can read off the value of g_{00} from Eq. (43)

$$g^{00} = \frac{1}{g_{00}} = \frac{1}{R^2\left(1 + 2\Phi/c^2\right)} \tag{47}$$

We will now compute only $\Gamma^0_{00}, \Gamma^0_{01}, \Gamma^0_{02}, \Gamma^0_{03}, \Gamma^0_{11}, \Gamma^0_{22}, \Gamma^0_{33}$, because we know that all others must be zero. We keep this to zero order in Φ.

Γ^0_{00}

$$\Gamma^0_{00} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{g_{00,0} + g_{00,0} - g_{00,0}\right\} \tag{48}$$

$$\Rightarrow \Gamma^0_{00} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{\frac{\partial}{\partial\eta}\left(R^2\left(1 + 2\Phi/c^2\right)\right)\right\} \tag{49}$$

$$\Rightarrow \Gamma^0_{00} = \frac{1}{2R^2}\frac{\partial\left(R^2\right)}{\partial\eta} = \frac{1}{2R^2}2R\frac{\partial R}{\partial\eta} = \frac{1}{R}\frac{\partial R}{\partial\eta} \tag{50}$$

Γ^0_{01}

$$\Gamma^0_{01} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{g_{00,1} + g_{01,0} - g_{01,0}\right\} \tag{51}$$

$$\Rightarrow \Gamma^0_{01} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{\frac{\partial}{\partial r}\left(R^2\left(1 + 2\Phi/c^2\right)\right)\right\} \tag{52}$$

$$\Rightarrow \Gamma^0_{01} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{\frac{\partial}{\partial r}\left(R^2\right)\right\} = \frac{R^2}{2R^2}\left\{\frac{\partial}{\partial r}(1)\right\} = 0 \tag{53}$$

Γ^0_{02}

$$\Gamma^0_{02} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{g_{00,2} + g_{02,0} - g_{02,0}\right\} \tag{54}$$

$$\Rightarrow \Gamma^0_{02} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{\frac{\partial}{\partial\theta_x}\left(R^2(1)\right)\right\} = 0 \tag{55}$$

Γ^0_{03}

$$\Gamma^0_{03} = \frac{1}{2R^2}\left(1 - \frac{2\Phi}{c^2}\right)\left\{g_{00,3}\right\} = \frac{1}{2R^2}\left\{\frac{\partial}{\partial\theta_y}\left(R^2(1)\right)\right\} = 0 \tag{56}$$

Γ^0_{11}

$$\Gamma^0_{11} = \frac{1}{2R^2}\left\{-g_{11,0}\right\} = \frac{-1}{2R^2}\left\{\frac{\partial}{\partial\eta}\left(-R^2\right)\right\} \tag{57}$$

$$\Rightarrow \Gamma^0_{11} = \frac{1}{2R^2}\frac{\partial\left(R^2\right)}{\partial\eta} = \frac{1}{2R^2}2R\frac{\partial R}{\partial\eta} = \frac{1}{R}\frac{\partial R}{\partial\eta} \tag{58}$$

Γ^0_{22}

$$\Gamma^0_{22} = \frac{1}{2R^2}\left\{-g_{22,0}\right\} = \frac{-1}{2R^2}\left\{\frac{\partial}{\partial\eta}\left(-R^2r^2\right)\right\} \tag{59}$$

$$\Rightarrow \Gamma^0_{22} = \frac{r^2}{2R^2}\frac{\partial\left(R^2\right)}{\partial\eta} = \frac{r^2}{R}\frac{\partial R}{\partial\eta} \tag{60}$$

Γ^0_{33}

$$\Gamma^0_{33} = \frac{1}{2R^2}\left\{-g_{33,0}\right\} = \frac{-1}{2R^2}\left\{\frac{\partial}{\partial\eta}\left(-R^2r^2\right)\right\} \tag{61}$$

$$\Rightarrow \Gamma^0_{33} = \frac{r^2}{2R^2}\frac{\partial\left(R^2\right)}{\partial\eta} = \frac{r^2}{R}\frac{\partial R}{\partial\eta} \tag{62}$$

So we return to Eq. (45) which can now be written as

$$\ddot{x}^0 + \Gamma^0_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = \ddot{x}^0 + \Gamma^0_{00}\dot{x}^0\dot{x}^0 + \Gamma^0_{11}\dot{x}^1\dot{x}^1 + \Gamma^0_{22}\dot{x}^2\dot{x}^2 + \Gamma^0_{33}\dot{x}^3\dot{x}^3 = 0 \tag{63}$$

$$\Rightarrow \ddot{x}^0 + \frac{1}{R}\frac{\partial R}{\partial\eta}\dot{x}^0\dot{x}^0 + \frac{1}{R}\frac{\partial R}{\partial\eta}\dot{x}^1\dot{x}^1 + \frac{r^2}{R}\frac{\partial R}{\partial\eta}\dot{x}^2\dot{x}^2 + \frac{r^2}{R}\frac{\partial R}{\partial\eta}\dot{x}^3\dot{x}^3 = 0 \tag{64}$$

$$\Rightarrow \frac{d^2\eta}{dp^2} + \frac{1}{R}\frac{\partial R}{\partial\eta}\left(\frac{d\eta}{dp}\right)^2 + \frac{1}{R}\frac{\partial R}{\partial\eta}\left(\frac{dr}{dp}\right)^2 + \frac{r^2}{R}\frac{\partial R}{\partial\eta}\left(\frac{d\theta_x}{dp}\right)^2 + \frac{r^2}{R}\frac{\partial R}{\partial\eta}\left(\frac{d\theta_y}{dp}\right)^2 = 0 \tag{65}$$

$$\Rightarrow \frac{d^2\eta}{dp^2} = -\frac{1}{R}\frac{\partial R}{\partial\eta}\left\{\left(\frac{d\eta}{dp}\right)^2 + \left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\theta_x}{dp}\right)^2 + r^2\left(\frac{d\theta_y}{dp}\right)^2\right\} \tag{66}$$

Now, from Eq. (44) we see that

$$\frac{dr}{dp} = \frac{dr}{d\eta}\frac{d\eta}{dp} = (-1)\,\frac{d\eta}{dp}\ , \tag{67}$$

so to first order we have

$$\frac{d^2\eta}{dp^2} = -\frac{1}{R}\frac{\partial R}{\partial\eta}\left\{\left(\frac{d\eta}{dp}\right)^2 + \left(\frac{d\eta}{dp}\right)^2\right\} = -\frac{2}{R}\frac{\partial R}{\partial\eta}\left(\frac{d\eta}{dp}\right)^2 = -\frac{2}{R}\frac{dR}{dp}\frac{d\eta}{dp}\ . \tag{68}$$

Therefore by choosing units of p appropriately, we find that

$$\boxed{\frac{d\eta}{dp} = -\frac{1}{R^2}\ .} \tag{69}$$

Now we will look at the $\mu = 2$ cases in Eq. (39):

$$\ddot{x}^2 + \Gamma^2_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\ , \tag{70}$$

non-zero affine connections have $\mu = \rho = 2$:

$$\Gamma^2_{\nu\sigma} = \frac{1}{2} g^{2\,2} \left\{ g_{\nu\;2,\sigma} + g_{2\;\sigma,\nu} - g_{\nu\sigma,2} \right\} \;. \tag{71}$$

We can read off the value of g_{22} from Eq. (43)

$$g^{22} = \frac{1}{g_{22}} = \frac{-1}{R^2 r^2 \left(1 - 2\Phi/c^2\right)} \tag{72}$$

We will now compute $\Gamma^2_{00}, \Gamma^2_{02}, \Gamma^2_{22}, \Gamma^2_{23}, \Gamma^2_{21}, \Gamma^2_{11}, \Gamma^2_{33}$, (we know that all others must be zero) to first order[4].

Γ^2_{00}

$$\Gamma^2_{00} = \left\{ \frac{-1}{2R^2 r^2 \left(1 - 2\Phi/c^2\right)} \right\} \left\{ -R^2 \frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \right\} = \frac{1}{r^2 c^2} \frac{\partial \Phi}{\partial \theta_x} \tag{73}$$

Γ^2_{02}

$$\Gamma^2_{02} = \left\{ \frac{-1}{2R^2 r^2 \left(1 - 2\Phi/c^2\right)} \right\} \left\{ \frac{d\left(R^2\right)}{d\eta} \left(-r^2 \left(1 - \frac{2\Phi}{c^2} \right) \right) \right\} \tag{74}$$

$$\Rightarrow \Gamma^2_{02} = \left\{ \frac{-1}{2R^2 r^2} \right\} \left\{ -2Rr^2 \frac{dR}{d\eta} \right\} = \frac{1}{R} \frac{dR}{d\eta} \tag{75}$$

Γ^2_{22}

$$\Gamma^2_{22} = \left\{ \frac{-1}{2R^2 r^2 \left(1 - 2\Phi/c^2\right)} \right\} \left\{ -R^2 r^2 \left(-\frac{2}{c^2} \frac{d\Phi}{d\theta_x} \right) \right\} = -\frac{\partial \Phi}{\partial \theta_x} \frac{1}{c^2} \tag{76}$$

Γ^2_{23}

$$\Gamma^2_{23} = \left\{ \frac{-1}{2R^2 r^2 \left(1 - 2\Phi/c^2\right)} \right\} \left\{ R^2 r^2 \frac{2}{c^2} \frac{d\Phi}{d\theta_y} \right\} = -\frac{\partial \Phi}{\partial \theta_x} \frac{1}{c^2} \tag{77}$$

Γ^2_{21}

$$\Gamma^2_{21} = \left\{ \frac{-1}{2R^2 r^2 \left(1 - 2\Phi/c^2\right)} \right\} \left\{ \frac{\partial}{\partial r} \left(-R^2 r^2 \left(1 - \frac{2\Phi}{c^2} \right) \right) \right\} \tag{78}$$

$$\Rightarrow \Gamma^2_{21} = \left\{ \frac{-1}{2R^2 r^2} \right\} \left\{ -2R^2 r \left(1 - 2\Phi/c^2 \right) - R^2 r^2 \left(-\frac{2}{c^2} \right) \frac{\partial \Phi}{\partial r}) \right\} \tag{79}$$

$$\Rightarrow \Gamma^2_{21} = \frac{1}{r} \left(1 - 2\Phi/c^2 \right) - \frac{1}{c^2} \frac{\partial \Phi}{\partial r} \tag{80}$$

Γ^2_{11}

$$\Gamma^2_{11} = \left\{ \frac{-1}{2R^2 r^2 \left(1 - 2\Phi/c^2\right)} \right\} \left\{ -\frac{2R^2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \right\} = \frac{1}{c^2 r^2} \frac{\partial \Phi}{\partial \theta_x} \tag{81}$$

[4]To zero order in Φ as before

Γ^2_{33}

$$\Gamma^2_{33} = \left\{\frac{-1}{2R^2r^2\left(1-2\Phi/c^2\right)}\right\}\left\{\frac{2R^2}{c^2r^2}\frac{\partial\Phi}{\partial\theta_x}\right\} = -\frac{1}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{82}$$

So we return to Eq. (70) which can now be written as

$$\ddot{x}^2 + \Gamma^2_{00}\dot{x}^0\dot{x}^0 + 2\Gamma^2_{02}\dot{x}^0\dot{x}^2 + \Gamma^2_{22}\dot{x}^2\dot{x}^2 + 2\Gamma^2_{23}\dot{x}^2\dot{x}^3 + 2\Gamma^2_{21}\dot{x}^2\dot{x}^1 + \Gamma^2_{11}\dot{x}^1\dot{x}^1 + \Gamma^2_{33}\dot{x}^3\dot{x}^3 = 0 \tag{83}$$

$$\begin{aligned}\Rightarrow \ddot{\theta}_x &+ \tfrac{1}{r^2c^2}\tfrac{\partial\Phi}{\partial\theta_x}\left(\tfrac{d\eta}{dp}\right)^2 + \tfrac{2}{R}\tfrac{dR}{d\eta}\tfrac{d\eta}{dp}\tfrac{d\theta_x}{dp} - \tfrac{\partial\Phi}{\partial\theta_y}\tfrac{1}{c^2}\left(\tfrac{d\theta_x}{dp}\right)^2 - \tfrac{\partial\Phi}{\partial\theta_x}\tfrac{2}{c^2}\tfrac{d\theta_x}{dp}\tfrac{d\theta_y}{dp}\\ &+\left(\tfrac{2}{r}\left(1-\tfrac{2\Phi}{c^2}\right) - \tfrac{1}{c^2}\tfrac{\partial\Phi}{\partial r}\right)\tfrac{dr}{dp}\tfrac{d\theta_x}{dp} + \tfrac{1}{c^2r^2}\tfrac{\partial\Phi}{\partial\theta_x}\left(\tfrac{dr}{dp}\right)^2 - \tfrac{1}{c^2}\tfrac{\partial\Phi}{\partial\theta_x}\left(\tfrac{d\theta_y}{dp}\right)^2 = 0\end{aligned} \tag{84}$$

Note that the factor of 2 arises in front of Γ^2_{02}, Γ^2_{23} and Γ^2_{21} due to the fact that we must include e.g. Γ^2_{02} *and* Γ^2_{20}. Keeping to first order in Φ, this reduces to

$$\ddot{\theta}_x + \frac{1}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x}\left(\frac{d\eta}{dp}\right)^2 + \frac{2}{R}\frac{dR}{d\eta}\frac{d\eta}{dp}\frac{d\theta_x}{dp} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta_x}{dp} + \frac{1}{c^2r^2}\frac{\partial\Phi}{\partial\theta_x}\left(\frac{dr}{dp}\right)^2 = 0 \tag{85}$$

$$\Rightarrow \ddot{\theta}_x + \frac{1}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x}\left(\frac{1}{R^2}\right)^2 + \frac{2}{R}\frac{dR}{d\eta}\frac{d\eta}{dp}\frac{d\theta_x}{dp} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta_x}{dp} + \frac{1}{c^2r^2}\frac{\partial\Phi}{\partial\theta_x}\left(-\frac{1}{R^2}\right)^2 = 0 \tag{86}$$

$$\boxed{\Rightarrow \ddot{\theta}_x + \frac{2}{r^2c^2R^4}\frac{\partial\Phi}{\partial\theta_x} + \frac{2}{R}\frac{dR}{dp}\frac{d\theta_x}{dp} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta_x}{dp} = 0 \qquad (87)}$$

Now we want to change from $\ddot{\theta}_x$ to $\frac{d^2\theta_x}{d\eta^2}$:

$$\frac{d^2\theta_x}{d\eta^2} = \frac{d}{d\eta}\left(\frac{d\theta_x}{d\eta}\right) = \frac{dp}{d\eta}\frac{d}{dp}\left(\frac{dp}{d\eta}\frac{d\theta_x}{dp}\right) = \left(\frac{dp}{d\eta}\right)^2\frac{d^2\theta_x}{dp^2} + \frac{dp}{d\eta}\frac{d\theta_x}{dp}\frac{d}{dp}\left(\frac{dp}{d\eta}\right) \tag{88}$$

and we can use the fact that

$$\frac{dr}{dp} = \frac{dr}{d\eta}\frac{d\eta}{dp} = -\frac{d\eta}{dp} \tag{89}$$

to give

$$\frac{d^2\theta_x}{d\eta^2} = R^4\frac{d^2\theta_x}{dp^2} + R^2\frac{d\theta_x}{dp}2R\frac{dR}{dp} \tag{90}$$

So substituting from Eq. (87) gives

$$\frac{d^2\theta_x}{d\eta^2} = R^4\left(-\frac{2}{r^2c^2R^4}\frac{\partial\Phi}{\partial\theta_x} - \frac{2}{R}\frac{dR}{dp}\frac{d\theta_x}{dp} - \frac{2}{r}\frac{dr}{dp}\frac{d\theta_x}{dp}\right) + 2R^3\frac{d\theta_x}{dp}\frac{dR}{dp} \tag{91}$$

$$\Rightarrow \frac{d^2\theta_x}{d\eta^2} = -\frac{2}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x} - 2R^3\frac{dR}{dp}\frac{d\theta_x}{dp} - \frac{2R^4}{r}\frac{dr}{dp}\frac{d\theta_x}{dp} + 2R^3\frac{d\theta_x}{dp}\frac{dR}{dp} \tag{92}$$

$$\Rightarrow \frac{d^2\theta_x}{d\eta^2} = -\frac{2}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x} - \frac{2R^4}{r}\frac{dr}{dp}\frac{d\theta_x}{dp} \tag{93}$$

$$\Rightarrow \frac{d^2\theta_x}{d\eta^2} + \frac{2R^4}{r}\frac{dr}{d\eta}\frac{d\eta}{dp}\frac{d\theta_x}{d\eta}\frac{d\eta}{dp} = -\frac{2}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{94}$$

$$\Rightarrow \frac{d^2\theta_x}{d\eta^2} + \frac{2R^4}{r}(-1)\frac{1}{R^2}\frac{d\theta_x}{d\eta}\frac{1}{R^2} = -\frac{2}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{95}$$

$$\boxed{\frac{d^2\theta_x}{d\eta^2} - \frac{2}{r}\frac{d\theta_x}{d\eta} = -\frac{2}{r^2c^2}\frac{\partial\Phi}{\partial\theta_x}} \tag{96}$$

You will often see this result expressed in terms of the comoving transverse distance, $X = S_k\theta_x$. We start by expressing $\dot{X}$ and $\ddot{X}$:

$$\dot{X} = \dot{S}_k\theta_x + S_k\dot{\theta}_x \tag{97}$$

$$\ddot{X} = \ddot{S}_k\theta_x + \dot{S}_k\dot{\theta}_x + \dot{S}_k\dot{\theta}_x + S_k\ddot{\theta}_x = \ddot{S}_k\theta_x + 2\dot{S}_k\dot{\theta}_x + S_k\ddot{\theta}_x \tag{98}$$

Recall the definition of S_k,

$$S_k(r) = \begin{cases} \sin r & (k=1) \\ \sinh r & (k=-1) \\ r & (k=0)\,. \end{cases} \tag{99}$$

Flat Case

We will now consider the flat case. In this case $S_k = r$ and so $\dot{S}_k = \dot{r} = -1/R^2$, and

$$\ddot{S}_k = \ddot{r} = \frac{d\eta}{dp}\frac{d}{d\eta}(\dot{r}) = \frac{1}{R^2}\frac{d}{d\eta}\left(-\frac{1}{R^2}\right) = \frac{1}{R^2}\left(2R^{-3}\frac{dr}{d\eta}\right) = \frac{2}{R^5}\frac{dR}{d\eta}\,. \tag{100}$$

In this case we can write $\dot{X}$ and $\ddot{X}$ as:

$$\dot{X} = \dot{S}_k\theta_x + S_k\dot{\theta}_x = -\frac{\theta_x}{R^2} + r\dot{\theta}_x = -\frac{X}{rR^2} + r\dot{\theta}_x \tag{101}$$

$$\Rightarrow \dot{\theta}_x = \frac{\dot{X}}{r} + \frac{X}{r^2R^2} \tag{102}$$

$$\ddot{X} = \ddot{S}_k\theta_x + 2\dot{S}_k\dot{\theta}_x + S_k\ddot{\theta}_x = \frac{2}{R^5}\frac{dR}{d\eta}\theta_x - \frac{2}{R^2}\dot{\theta}_x + r\ddot{\theta}_x = \frac{2}{R^5}\frac{dR}{d\eta}\frac{X}{r} - \frac{2}{R^2}\dot{\theta}_x + r\ddot{\theta}_x \tag{103}$$

$$\Rightarrow \ddot{\theta}_x = \frac{\ddot{X}}{r} + \frac{2}{rR^2}\dot{\theta}_x - \frac{2}{r^2R^5}\frac{dR}{d\eta}X \tag{104}$$

Now we substitute for $\dot{\theta}_x$ from Eq. (102):

$$\Rightarrow \ddot{\theta}_x = \frac{\ddot{X}}{r} + \frac{2}{rR^2}\left(\frac{\dot{X}}{r} + \frac{X}{r^2R^2}\right) - \frac{2}{r^2R^5}\frac{dR}{d\eta}X \tag{105}$$

$$\Rightarrow \ddot{\theta}_x = \frac{\ddot{X}}{r} + \frac{2\dot{X}}{r^2R^2} + \frac{2X}{r^3R^4} - \frac{2}{r^2R^5}\frac{dR}{d\eta}X \tag{106}$$

Now returning to Eq. (87), we can use Eq. (102) and Eq. (106) to re-write it as:

$$\frac{\ddot{X}}{r} + \frac{2\dot{X}}{r^2R^2} + \frac{2X}{r^3R^4} - \frac{2}{r^2R^5}\frac{dR}{d\eta}X + \frac{2}{r^2c^2R^4}\frac{\partial\Phi}{\partial\theta_x} + \frac{2}{R}\dot{R}\left(\frac{\dot{X}}{r} + \frac{X}{r^2R^2}\right) + \frac{2}{r}\dot{r}\left(\frac{\dot{X}}{r} + \frac{X}{r^2R^2}\right) = 0 \quad (107)$$

$$\Rightarrow \frac{\ddot{X}}{r} + \left(\frac{2}{r^2R^2} + \frac{2}{rR}\dot{R} + \frac{2}{r^2}\dot{r}\right)\dot{X} + \left(-\frac{2}{r^2R^5}\frac{dR}{d\eta} + \frac{2}{r^3R^4} + \frac{2}{r^2R^3}\dot{R} + \frac{2}{r^3R^2}\dot{r}\right)X = -\frac{2}{r^2c^2R^4}\frac{\partial\Phi}{\partial\theta_x} \quad (108)$$

$$= \frac{\ddot{X}}{r} + \left(\frac{2}{r^2R^2} + \frac{2}{rR}\frac{dR}{d\eta}\frac{1}{R^2} + \frac{2}{r^2}\left(-\frac{1}{R^2}\right)\right)\dot{X} + \left(-\frac{2}{r^2R^5}\frac{dR}{d\eta} + \frac{2}{r^3R^4} + \frac{2}{r^2R^3}\frac{dR}{d\eta}\frac{1}{R^2} + \frac{2}{r^3R^2}\left(-\frac{1}{R^2}\right)\right)X \quad (109)$$

$$\Rightarrow \ddot{X} + \frac{2}{R}\frac{dR}{d\eta}\frac{1}{R^2}\dot{X} = -\frac{2}{rc^2R^4}\frac{\partial\Phi}{\partial\theta_x} \quad (110)$$

Now we use the fact that:

$$\frac{d^2X}{dp^2} = \frac{d}{dp}\frac{dX}{dp} = \frac{d\eta}{dp}\frac{d}{d\eta}\left(\frac{dX}{d\eta}\frac{d\eta}{dp}\right) = \frac{1}{R^2}\frac{d}{d\eta}\left(\frac{dX}{d\eta}\frac{1}{R^2}\right) \quad (111)$$

$$\Rightarrow \frac{d^2X}{dp^2} = \frac{1}{R^2}\left(\frac{d^2X}{d\eta^2}\frac{1}{R^2} + \frac{dX}{d\eta}\frac{d}{d\eta}\left(\frac{1}{R^2}\right)\right) = \frac{1}{R^4}\frac{d^2X}{d\eta^2} + \frac{1}{R^2}\frac{dX}{d\eta}\left(-2R^{-3}\frac{dR}{d\eta}\right) \quad (112)$$

Lastly, we return to Eq. (110), substituting $\ddot{\theta}_x$

$$\frac{1}{R^4}\frac{d^2X}{d\eta^2} - \frac{2}{R^5}\frac{dX}{d\eta}\frac{dR}{d\eta} + \frac{2}{R^3}\frac{dR}{d\eta}\frac{1}{R^2}\frac{dX}{d\eta} = -\frac{2}{rc^2R^4}\frac{\partial\Phi}{\partial\theta_x} \quad (113)$$

$$\Rightarrow \frac{d^2X}{d\eta^2} = -\frac{2}{c^2r}\frac{\partial\Phi}{\partial\theta_x} \quad (114)$$

Finally this allows us to write the result for the flat case:

$$\boxed{\frac{d^2\boldsymbol{X}}{d\eta^2} = -\frac{2}{c^2}\boldsymbol{\nabla}_\perp\Phi\ . \quad (115)}$$

where $\boldsymbol{\nabla}_\perp = \left(\frac{d}{dX}, \frac{d}{dY}\right)$.

B Appendix

B.1 Derivation of the deflection angle from the Euler-Lagrange Equation

A more compact way to derive the deflection angle is to use the Euler-Lagrange equation (again a dot denotes d/dp):

$$\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = 0 \quad (116)$$

We are free to choose L^2 in the Euler-Lagrange equation and we set it to be equal to $(ds/dp)^2$,

$$L^2 = R^2(\eta)\left\{\left(1 + \frac{2\Phi}{c^2}\right)\dot{\eta}^2 - \left(1 - \frac{2\Phi}{c^2}\right)\left[\dot{r}^2 + S_k^2\left(\dot{\theta}_x^2 + \dot{\theta}_y^2\right)\right]\right\}\ , \quad (117)$$

and where we define $x^\mu = (\eta, r, \theta_x, \theta_y)$. We now calculate Eq. (116) for the different values of μ. These are calculated to zero order in Φ.

$\mu = 0$

$$\frac{\partial L^2}{\partial \eta} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{\eta}}\right) = 0 \tag{118}$$

$$\Rightarrow 2R\frac{dR}{d\eta}\left\{\left(1+\frac{2\Phi}{c^2}\right)\dot{\eta}^2 - \left(1-\frac{2\Phi}{c^2}\right)\left[\dot{r}^2 + S_k^2\left(\dot{\theta}_x^2 + \dot{\theta}_y^2\right)\right]\right\} - \frac{d}{dp}\left[R^2 2\dot{\eta}\right] = 0 \tag{119}$$

For photons we know that $ds^2 = 0$ therefore $L^2 = 0$ and so the term

$$\left\{\left(1+\frac{2\Phi}{c^2}\right)\dot{\eta}^2 - \left(1-\frac{2\Phi}{c^2}\right)\left[\dot{r}^2 + S_k^2\left(\dot{\theta}_x^2 + \dot{\theta}_y^2\right)\right]\right\} = 0\ . \tag{120}$$

Therefore we can write

$$-\frac{d}{dp}\left[R^2 2\dot{\eta}\right] = 0 \tag{121}$$

$$\Rightarrow R^2\dot{\eta} = \text{constant} \tag{122}$$

$$\Rightarrow \frac{d\eta}{dp} = \frac{1}{R^2} \text{ by choice of suitable units of p.} \tag{123}$$

$\mu = 1$

$$\frac{\partial L^2}{\partial r} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{r}}\right) = 0 \tag{124}$$

To zeroth order, for an incoming, radial ray we need only use the fact that $ds^2 = 0$ to give straightforwardly

$$0 = R^2\left[d\eta^2 - dr^2\right]\ , \tag{125}$$

$$\Rightarrow \frac{dr}{d\eta} = -1\ , \tag{126}$$

$$\Rightarrow \dot{r} = \frac{dr}{dp} = \frac{dr}{d\eta}\frac{d\eta}{dp} = \frac{(-1)}{R^2}\ . \tag{127}$$

$\mu = 2$

$$\frac{\partial L^2}{\partial \theta_x} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{\theta}_x}\right) = 0 \tag{128}$$

$$\Rightarrow R^2\dot{\eta}^2\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} + \frac{2R^2}{c^2}\frac{\partial\Phi}{\partial\theta_x}\left[\dot{r}^2 + S_k^2(r)\left(\dot{\theta_x}^2 + \dot{\theta_y}^2\right)\right] - \frac{d}{dp}\left(-R^2S_k^2 2\dot{\theta}_x\left(1-\frac{2\Phi}{c^2}\right)\right) = 0 \tag{129}$$

Ignoring terms $\frac{\partial\Phi}{\partial\theta_x}\left(\dot{\theta_x}^2 + \dot{\theta_y}^2\right)$ and $\dot{\theta}_x\left(-\frac{2\Phi}{c^2}\right)$ as they are 2nd order, we now write

$$\Rightarrow R^2\dot{\eta}^2\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} + \frac{2R^2}{c^2}\frac{\partial\Phi}{\partial\theta_x}\dot{r}^2 - \frac{d}{dp}\left(-R^2S_k^2 2\dot{\theta}_x\right) = 0 \tag{130}$$

Substituting in from Eq. (123) and Eq. (127) we get

$$R^2 \frac{1}{R^4} \frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} + \frac{2R^2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \frac{1}{R^4} + \frac{d}{dp} \left(2R^2 S_k^2 \dot{\theta}_x \right) = 0 \tag{131}$$

$$\Rightarrow \frac{2}{c^2 R^2} \frac{\partial \Phi}{\partial \theta_x} + \frac{2}{c^2 R^2} \frac{\partial \Phi}{\partial \theta_x} + 2 \frac{d}{dp} \left(R^2 S_k^2 \dot{\theta}_x \right) = 0 \tag{132}$$

$$\Rightarrow \frac{4}{c^2 R^2} \frac{\partial \Phi}{\partial \theta_x} = -2 \frac{d}{dp} \left(R^2 S_k^2 \dot{\theta}_x \right) \tag{133}$$

$$\Rightarrow R^2 \frac{d}{dp} \left(R^2 S_k^2 \dot{\theta}_x \right) = -\frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \tag{134}$$

We now want an expression for $d^2X/d\eta^2$, where $X = S_k \theta_x$ is the comoving transverse distance. We start by expressing $\dot{X}$:

$$\dot{X} = \dot{S}_k \theta_x + S_k \dot{\theta}_x \tag{135}$$

Recall the definition of S_k,

$$S_k(r) = \begin{cases} \sin r & (k = 1) \\ \sinh r & (k = -1) \\ r & (k = 0)\ . \end{cases} \tag{136}$$

Flat Case

Firstly we will consider the flat case. In this case $S_k = r$ and so $\dot{S}_k = \dot{r} = -1/R^2$. Therefore Eq. (135) becomes

$$\dot{X} = \frac{\theta_x}{R^2} + r\dot{\theta}_x = \frac{-X}{rR^2} + r\dot{\theta}_x \tag{137}$$

Now we can substitute for $\dot{\theta}_x$ in Eq. (134):

$$R^2 \frac{d}{dp} \left(R^2 r^2 \left[\frac{\dot{X}}{r} + \frac{X}{r^2 R^2} \right] \right) = -\frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \tag{138}$$

Now we want an expression for d/dp:

$$\frac{d}{dp} = \frac{d\eta}{dp} \frac{d}{d\eta} = \frac{1}{R^2} \frac{d}{d\eta} \tag{139}$$

$$\Rightarrow \frac{R^2}{R^2} \frac{d}{d\eta} \left(R^2 r^2 \left[\frac{1}{r} \frac{1}{R^2} \frac{dX}{d\eta} + \frac{X}{r^2 R^2} \right] \right) = -\frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \tag{140}$$

$$\Rightarrow \frac{d}{d\eta} \left(r \frac{dX}{d\eta} + X \right) = -\frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \tag{141}$$

$$\Rightarrow \frac{dr}{d\eta} \frac{dx}{d\eta} + r \frac{d^2 X}{d\eta^2} + \frac{dX}{d\eta} = -\frac{2}{c^2} \frac{\partial \Phi}{\partial \theta_x} \tag{142}$$

Recall that we learnt in Eq. (126) that $\frac{dr}{d\eta} = -1$, therefore

$$\frac{d^2 X}{d\eta^2} = -\frac{2}{c^2 r} \frac{\partial \Phi}{\partial \theta_x} \tag{143}$$

Finally this allows us to write the result for the flat case:

$$\boxed{\frac{d^2\boldsymbol{X}}{d\eta^2} = -\frac{2}{c^2}\boldsymbol{\nabla}_\perp\Phi\ . \qquad (144)}$$

where $\boldsymbol{\nabla}_\perp = \left(\frac{d}{dX}, \frac{d}{dY}\right)$.

General Case

Now we will look at the General case, firstly by finding an expression for $\dot{S}_k$:

$$S_k(r) = \begin{cases} \sin r \\ \sinh r \\ r\ , \end{cases} \tag{145}$$

$$\Rightarrow \dot{S}_k(r) = \begin{cases} \cos r\ \dot{r} \\ \cosh r\ \dot{r} \\ \dot{r} \end{cases} = \begin{cases} \left(1-\sin^2 r\right)^{\frac{1}{2}}\dot{r} \\ \left(1+\sinh^2 r\right)^{\frac{1}{2}}\dot{r} \\ \dot{r} \end{cases} = \begin{cases} \left(1-s_k^2\right)^{\frac{1}{2}}\dot{r} \\ \left(1+s_k^2\right)^{\frac{1}{2}}\dot{r} \\ \dot{r} \end{cases} = \left(1-kS_k^2\right)^{\frac{1}{2}}\dot{r} \tag{146}$$

$$\Rightarrow \dot{S}_k(r) = -\frac{\left(1-kS_k^2\right)^{\frac{1}{2}}}{R^2}\ . \tag{147}$$

We will substitute into Eq. (134) and then in to Eq. (135).

$$\dot{X} = \dot{S}_k\theta_x + S_k\dot{\theta}_x \tag{148}$$

$$\Rightarrow \dot{X} = -\frac{\left(1-kS_k^2\right)^{\frac{1}{2}}}{R^2}\theta_x + S_k\dot{\theta}_x \tag{149}$$

$$\Rightarrow \dot{X} = -\frac{X}{S_kR^2}\left(1-kS_k^2\right)^{\frac{1}{2}} + S_k\dot{\theta}_x \tag{150}$$

$$\Rightarrow \dot{\theta}_x = \frac{\dot{X}}{S_k} + \left(1-kS_k^2\right)^{\frac{1}{2}}\frac{X}{R^2S_k^2} \tag{151}$$

Now we substitute this expression for $\dot{\theta}_x$, and that for d/dp from Eq. (139) into Eq. (134):

$$\frac{d}{d\eta}\left(R^2S_k^2\left[\frac{dX}{dp}\frac{1}{S_k} + \left(1-kS_k^2\right)^{\frac{1}{2}}\frac{X}{R^2S_k^2}\right]\right) = -\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{152}$$

$$\Rightarrow \frac{d}{d\eta}\left(S_k\frac{dX}{d\eta} + X\left(1-kS_k^2\right)^{\frac{1}{2}}\right) = -\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{153}$$

$$\Rightarrow \frac{dS_k}{d\eta}\frac{dX}{d\eta} + S_k\frac{d^2X}{d\eta^2} + \left(1-kS_k^2\right)^{\frac{1}{2}}\frac{dX}{d\eta} + \frac{1}{2}X\left(1-kS_k^2\right)^{-\frac{1}{2}}\left(-2kS_k\frac{dS_k}{d\eta}\right) = -\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{154}$$

$$\Rightarrow \frac{dp}{d\eta}\frac{dS_k}{dp}\frac{dX}{d\eta} + S_k\frac{d^2X}{d\eta^2} + \left(1-kS_k^2\right)^{\frac{1}{2}}\frac{dX}{d\eta} - \frac{kS_kX}{\left(1-kS_k^2\right)^{\frac{1}{2}}}\frac{dp}{d\eta}\frac{dS_k}{dp} = -\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{155}$$

Now substituting in for $\frac{dS_k}{dp}$ from Eq. (147)

$$R^2\left(-\frac{\left(1-kS_k^2\right)^{\frac{1}{2}}}{R^2}\right)\frac{dX}{d\eta}+S_k\frac{d^2X}{d\eta^2}+\left(1-kS_k^2\right)^{\frac{1}{2}}\frac{dX}{d\eta}-\frac{kS_kX}{\left(1-kS_k^2\right)^{\frac{1}{2}}}R^2\left(-\frac{\left(1-kS_k^2\right)^{\frac{1}{2}}}{R^2}\right)=-\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{156}$$

$$\Rightarrow -\left(1-kS_k^2\right)^{\frac{1}{2}}\frac{dX}{d\eta}+S_k\frac{d^2X}{d\eta^2}+\left(1-kS_k^2\right)^{\frac{1}{2}}\frac{dX}{d\eta}+\frac{kS_kX\left(1-kS_k^2\right)^{\frac{1}{2}}}{\left(1-kS_k^2\right)^{\frac{1}{2}}}=-\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{157}$$

$$\Rightarrow S_k\frac{d^2X}{d\eta^2}+kS_kX=-\frac{2}{c^2}\frac{\partial\Phi}{\partial\theta_x} \tag{158}$$

which leads us to the result for the general case:

$$\boxed{\frac{d^2\boldsymbol{X}}{d\eta^2}+k\boldsymbol{X}=-\frac{2}{c^2}\boldsymbol{\nabla}_\perp\Phi\,, \qquad (159)}$$

where $\boldsymbol{\nabla}_\perp=\left(\frac{d}{dX},\frac{d}{dY}\right)$.