Planet formation: Theories and Modelling techniques

Currently there are two basic models of planet formation; the planetesimal hypothesis and the theory of gravitational collapse. Both of these models use as their building blocks the material left-over from star formation; a protostar and surrounding nebular disk.

Planetesimal hypothesis

As the infall from the surrounding cloud is slowed and halted, the decreased accretion rate allows the nebular disk to cool. At lower temperatures, condensates can form out of the disk. Iron and silicate compounds, with their higher evaporation temperatures, condense first. At lower temperatures, towards the outer portions of the disk, ices can also form. In these regions ISM grains which were not evaporated in the heat of the star formation process also exist. Modelling the grain growth by agglomeration is difficult as there are several complex feedback mechanisms involved, including aerodynamical drag, thermal structure coupled to changing opacities in the disk, the charged state of the medium and coupling to magnetic fields. The interactions and relative importance of these mechanisms is poorly understood.

Relative velocities

Collisions between particles occur because they are moving relative to one another. In laminar nebulae the relative velocities have several sources:
(In the following equations Ω is the Keplerian angular frequency, T is the temperature of the particles, $\mu = (m_1 m_2)/(m_1 + m_2)$ where m_1, m_2 are the masses of the two particles, τ_{f1}, τ_{f2} are the friction times, $\tau_f = max(\tau_{f1}, \tau_{f2})$, z is the vertical distance from the midplane,ΔV is the velocity difference between the Keplerian velocity and rotational gas velocity, u_s, t_s are the velocity and turnover times of the smallest turbulent eddy and u_l, t_l are the same for the largest turbulent eddy.)
Brownian motion;
This is the random walk motion of small particles suspended in a fluid (gas) due to bombardment by molecules obeying a Maxwellian distribution.

$$\Delta v_{th} = \sqrt{8kT/\pi\mu}.$$

Sedimentation;
We can calculate friction times for particles (also known as stopping times)

which is the response time of the particle to the force exerted by gas drag, as

$$\tau_f = mV/F_d$$

where F_d is the drag force and depends on; the Reynolds number (a dimensionless parameter, the ratio of the inertial to viscous forces), the Knudsen number (defined as λ/d, λ is the mean free path of the gas molecules, d is the diameter of the particle), and on V. If we assume a hydrostatic equilibrium of gas particles then the vertical acceleration of the particle must be a balance of gravity against drag support from the gas.

$$\Delta v_s = \Omega^2 z|\tau_{f1} - \tau_{f2}|.$$

Radial drift;
Solar gravity effects a particle's motion on a timescale of $1/\Omega$. If $\tau_f \ll 1/\Omega$ then the drag force is dominant and the particle moves with the angular frequency of the gas. As the particle is not supported by the pressure gradient in the gas, there is a residual force in the radial direction and the particle drifts inwards at a terminal velocity of

$$v_r = -2\Omega\tau_{fi}\Delta V.$$

If we consider the relative motions of two particles then we have

$$\Delta v_r = 2\Omega\Delta V|\tau_{f1} - \tau_{f2}|.$$

If alternatively $\tau_f > 1/\Omega$ (larger bodies) then the particles move with Keplerian angular frequency and feel a head-wind from the slower moving gas disk. The radial decay then becomes

$$v_r = \frac{2\Delta V}{\tau_f\Omega}.$$

Turbulent motions;
In turbulent systems turbulence-driven motion can also be important. Original work by Völk et al (1980) considered the interactions between grains and individual turbulent eddies, assuming a Kolmogorov law for the turbulent viscosity in the gas, and the overall effect was taken to be the sum of all interactions with eddies smaller then the scale of the turbulence.

$$\langle\Delta v_{tur}\rangle = 1.33u_l\sqrt{\tau_f/t_l} \text{ for } t_l > \tau_{f1}, \tau_{f2} > t_s.$$

In this summation the inner scale of the turbulence was assumed to be zero, but in fact this scale is finite and given by the scale at which random motions of the gas decay due to molecular viscosity. Eddies of a size smaller than this do not exist as gas flow on this scale is laminar and therefore doesn't contribute to the random velocity. Weidenschilling considered that the forming (decaying) smallest eddies dominantly accelerate (decelerate) the laminar flow inducing a uniform frictional acceleration (deceleration) in the very small grains. From an order of magnitude estimate he found

$$\Delta v_{tur} \simeq u_S/t_s|\tau_{f1} - \tau_{f2}| \text{ for } \tau_{f1}, \tau_{f2} < t_s.$$

Growth rates - the particle in a box approximation

The growth process from microscopic to macroscopic sized grains is dependant on the sticking probabilities and the strength of the adhesion forces holding the aggregates together. The sticking probability depends on the collisional velocity, masses and shapes of the particles, and material properties. It is generally assumed that small particles are stuck together by van der Waals forces. The growth during this stage is by BCCA - ballistic cluster-cluster aggregation whereby clusters collide and stick without fragmentation or restructuring, leading to fluffy aggregates, or by BPCA - ballistic particle-cluster aggregation in which clusters grow by the addition of individual particles. This builds up the solid material from sub-micron sized to centimeter sized bodies and larger whilst the material settles towards the mid-plane. The increased densities in the midplane then increase the collisional rates and hence growth rates of the bodies.

To calculate the accretion rate of a given body in the early protoplanetary disk, the simplest methods ignore entirely the motion of bodies around the star, and concentrate instead on the motions relative to one another. The velocity at which two bodies collide is

$$v_{coll} = (v^2 + v_e^2)^{1/2}$$

where v is their relative velocity and v_e is the escape velocity from the point of contact, given by

$$v_e = \left(2G\frac{m_1 + m_2}{R_1 + R_2}\right),$$

where R_1, R_2 are the radii of the bodies involved. The rebound velocity is ϵv_c, where $\epsilon \leq 1$ is the coefficient of restitution. If $\epsilon v_c \leq v_e$ then the two bodies remain gravitationally bound and will soon re-collide and accrete. When considering the largest body in a swarm of planetesimals, the escape velocity is often higher than the relative velocities, and so the largest body will accrete the vast majority if the bodies with which it collides. We assume the body we are interested in is embedded in a uniform surface density and velocity dispersion disk of bodies. A collision occurs when the separation between two bodies becomes less than the sum of their radii, $R_s = R_1 + R_2$. The accretion rate for a body embedded in a swarm of planetesimals of density ρ_{sw} at a relative velocity v to the average velocity of the swarm is given by

$$\frac{dM}{dt} = \rho_{sw} v \pi R_s^2 \left[1 + \left(\frac{v_e}{v}\right)^2\right].$$

Here the $\left(\frac{v_e}{v}\right)^2$ term represents the gravitational enhancement of the collisional cross-section. This measures the increased chances of collision due to the gravitational attraction for the body under consideration. As the mass of the body and thus the escape velocity increases, the rate of growth is similarly increased.

Modelling disk structure

Simulations of planetesimal growth often employ a 1-d representation of the disk. 1-D radial modelling assumes that the midplane can be used to represent the disk. In this approximation the surface density of the disk is modelled as

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(\Sigma \nu r^{1/2})\right].$$

Here

$$\Sigma = \int_0^H \rho dz.$$

This method of modelling the disk structure can be combined with modelling dust aggregation through collisions etc (see, e.g. Schmitt et al 1997). Here the mass distribution of the dust in each radial box is calculated by models of collisional growth, and the opacity of the dust calculated from this. The opacity is then used to calculate the energetic structure of the disk, controlling disk evolution for the next timestep, and so on. The optical depth from midplane to surface of the disk is given by

$$\tau = \frac{1}{2}\Sigma \kappa_R$$

where $\kappa_R(\rho, T)$ is the Rosseland mean opacity,

$$\frac{1}{\langle \kappa \rangle} = \frac{1}{B}\int_0^\infty \frac{B_\nu}{\kappa_\nu} d\nu.$$

The opacity of the solid disk depends on the assumed material properties - the refractive indices and fractal structure of the bodies, which can be combined according to effective medium theory to calculate the opacity of the grains (see e.g. works by Ossenkopf).

The problem with modelling the disk structure in this way is that you are unable to treat the formation of a sub-disk by settling to the midplane as the disk is approximated in the radial direction.

Alternative models also in 1-d use a vertical description of disk structure (see e.g. the works of Weidenschilling). The temperature gradient normal to the plane is

$$\frac{dT}{dz} = -\frac{\mu g}{C_p}$$

where μ is the molecular weight, C_p is the specific heat at constant pressure and g is the component of stellar gravity in the vertical direction. Assuming we can neglect self-gravity as stellar gravity is much greater, $g = \Omega^2 z$. Integrating we find that temperature varies with z as

$$T = T_c\left(1 - \frac{z^2}{H^2}\right).$$

T_c is the value of T in the central plane $z = 0$, and H is the characteristic half thickness of the disk,

$$H = \left(\frac{2C_p T_c}{\mu \Omega^2}\right).$$

In an ideal adiabatic gas density varies as $T^{1/(\gamma-1)}$ where γ is the ratio of specific heats at constant pressure and volume, thus

$$\rho(z) = \rho_c\left(1 - \frac{z^2}{h^2}\right)^{1/(\gamma-1)}.$$

Then the half-surface density is given by

$$\Sigma = \rho_c \int_0^H \left(1 - \frac{z^2}{H^2}\right)^{1/(\gamma-1)} dz.$$

In these models the region above the midplane is divided into a series of levels. At each level the size distribution is divided into a series of bins; collision rates and growth are modelled using the particle-in-a-box approach. At each time step the change in population for each mass bin is calculated, and then turbulent diffusion and settling rates are computed for the new size distribution. These are used to redistribute mass between the vertical levels. This is similar to the methods used by the radial 1-d modelling, but in this case rather than assuming all matter to be in the midplane, we take only one radial location into consideration. The advantage of this technique is that settling rates, which are faster then radial drift rates, are incorporated fully. However, radial transport does become important over the timescale of dust settling, and thus 2 or even 3d modelling is necessary for an adequate description of dust dynamics. The computational expense of such models has meant that until now, very few studies of this nature have been carried out.

Runaway growth and the isolation mass

It can be seen that when the relative velocity between bodies is comparable to or larger than the escape velocity, the growth rate is $\propto R_s^2$. If the relative velocity is small in comparison to the escape velocity, the gravitational enhancement factor becomes $\propto 1 + v_e^2$ and as $v_e = \left(\frac{2G(m_1+m_2)}{R_s}\right)^{1/2}$, and $m \propto volume \propto R^3 \rightarrow v_e \propto R$ and we get $\frac{dM}{dt} \propto R_s^4$. In this situation the largest body will grow much more rapidly than other bodies in the swarm, and we have a process of runaway growth leading to a bimodal population distribution. A physical mechanism which could lead to lower relative velocities for larger bodies would be the increased gas drag experienced due to the gravitational concentration of nebula gas around these bodies.

Runaway accretion requires low random velocities, and therefore small radial excursions of the planetesimals ($2ae$). This implies that the runaway protoplanet's feeding zone is limited to an annulus of planetesimals which it can gravitationally perturb into intersecting orbits. Thus once most of the planetesimals which can be reached by the protoplanet have been consumed, runaway growth must stop.

For a protoplanet on an initially circular orbit, the restricted three body problem places a limit on the radius within which planetesimals can be perturbed into crossing orbits of 4 Hill's radii. The mass of a protoplanet that has accreted all of the planetesimals within an annulus of width $2\Delta r$ is

$$M = \int_{r-\Delta r}^{r+\Delta r} 2\pi r\prime \sigma(r\prime) dr\prime \approx 4\pi r \Delta r \sigma(r).$$

Setting $\Delta r = 4R_H$ gives the so-called isolation mass of the protoplanet - the maximum mass that can be reached if no additional mass can diffuse into the protoplanets accretion zone. Mechanisms which will allow such diffusion are scattering between planetesimals, perturbations by planetary embryos in neighbouring accretion zones and gas drag. Alternatively, gravitational torques resulting from the excitation of spiral density waves in the gaseous component of the disk have the potential of inducing radial migration of the protoplanet, thus moving it into an accretion zone not depleted of planetesimals. The gravitational focussing of the gas and resultant increased gas drag could also be important here.

Later stages of growth

At the end of the runaway growth period, most mass is concentrated into the planetary embryos. Thus the random velocities are no longer damped by energy equipartition with smaller planetesimals. Mutual gravitational scattering pumps up relative velocities of the protoplanets to levels comparable to the escape velocities. The continued growth and evolution of the bodies is now governed by close gravitational encounters and violent highly inelastic collisions. N-body codes can be used here to model the growth of terrestrial planets from the population of protoplanets. Such models have largely ignored the smaller planetesimals present entirely, and often result in a small number of bodies on high eccenticity orbits (compared to the solar system). It has been suggested that the remaining gravitational dampening effects of the remaining small bodies could damp the eccentricities and result in fewer collisions and a larger population of terrestrial planets in the final system. N-body codes are still computationally very expensive, especially over the timescales required by planetary formation scenarios. Alternatively, final stages of terrestrial planet formation can be modeled using Monte-Carlo techniques, something that has been developed extensively by Wetherill. This assumes that planetesimals interact according to slowly precessing orbits that are occasionally altered by rare close encounters with other bodies. The Monte-Carlo method is used to choose successive pairs of planetesimals that interact according to the two-body gravitational scattering formula. When the distance between two bodies becomes less than their physical radii summed, the bodies are assumed to merge to form a more massive body. Otherwise an encounter between two bodies results in gravitational perturbation to new orbits. If gravitational perturbations are strong, orbital crossing is maintained until only a few bodies remain. If dampening effects are stronger, then the bodies become isolated more quickly. Wetherill has used a wide variety of initial conditions in his simulations, but the results are surprisingly similar regardless of the end result of the proto-planet building stage.

Gas accretion

Proto-planets of mass $M \geq 0.1 M_\oplus$ have surface escape velocities which are larger than the sound speed in gaseous protoplanetary disks (based on observations and simulations). The mass of a proto-giant's core increases rapidly during the runaway stage. The temperature and thermal pressure increase. The escape velocity at the surface of the core is larger than the sound speed in the gaseous disk, and so gas falls onto the proto-planet. As the feeding zone is depleted temperature and thermal pressure decrease due to decreased accretion rate and the rate of accumulation of gas increases. When the gas mass becomes comparable to the solid mass of the proto-planet there begins a runaway gas accretion phase. The proto-planet may clear a gap in the disk and thus this runaway accretion stops. Alternatively, the planet may accumulate all of the gas in its region and end runaway in this way. The planet then contracts on Kelvin-Helmholtz timescales ($t_{KH} = E_g/L \sim GM^2/RL$). The temperature in the gas envelope increases rapidly during contraction so that the luminosity of the planet remains almost constant. Increasing incompressibility of the fluid envelope slows down the contraction, and thus temperature and luminosity decrease with time.

Gravitational collapse

The traditional view

Solid planetesimals drift towards the midplane on sedimentation timescales - forming a relatively thin disk. Assuming that flow is laminar in the disk, there would be negligible random velocities. This thin sheet would then be unstable to fragmentation and gravitational collapse. The threshold of stability can be seen by looking at the dispersion relation for local axisymmetric perturbations.

$$\omega^2 = k^2c^2 + \kappa^2 - 2\pi G\sigma k.$$

In a protoplanetary disk where the disk mass is dominated by the solid particles - as is the case assuming a thin disk after settling - $\kappa = \Omega$, the Keplerian angular frequency, G is the gravitational constant, c is used to parameterise random motions, ω is the frequency of the disturbance, σ is the surface density of the disk and $k = 2\pi/\lambda$ is the wavenumber of the disturbance. Now we assume that $c = 0$ as in the case of laminar flow, and show that if $\lambda > \lambda_c$, where $\lambda_c = \frac{4\pi^2 G\sigma_p}{\Omega^2}$ and σ_p is the surface density of the particles, then ω^2 becomes negative and the system is unstable to fragmentation. However, we can recall that the motion of the gas is slower than that of the solids. As the solid bodies drift to the midplane, their concentration raises the solid density : gas density ratio in the midplane. This forces gas trapped in the midplane to move at nearer the Keplerian velocity of the solids as the mean density is dominated by that of the solids and bulk coupling has occurred. This gives rise to large shears in the gas at near the midplane, inducing shear driven turbulence. This reduces the precipitation of solids and keeps the midplane thick enough that the disk cannot become unstable to the gravitational collapse described above, as random velocities will not be negligible. Particles must grow to at least meter sized before they can decouple from shear-induced turbulence and slow to the low velocities required for the instability to occur. The radial velocity dispersion induced by drag will further delay the onset of instability until the mean size is in the range of 10-100m.

New models of gravo-turbulent collapse

Recent work on gravo-turbulent collapse has focussed on the idea that, rather than being a hindrance to instability, turbulence in the disk actually triggers density clumping and thus enhances the chances of forming planetesimals by gravitational collapse (see e.g. Johansen, 2006). The turbulence in the midplane causes small (a few percent over the mean as turbulence is highly sub-sonic) over densities in the gas. These over densities are the locations of local pressure maxima. This pressure acts as an additional gravitational pull from the star for the disk interior to the maxima, creating a super-Keplerian flow. For the disk exterior to the maxima, the pressure acts to effect a decrease in the stellar gravity, leading to a sub-Keplerian flow. As the solids don't feel the pressure gradient and are moving at a Keplerian rate they are forced into the pressure bump - moving at Keplerian rates. The pressure over density in the gas is small and is therefore quick to disperse - typically lasting just a few orbits in simulations using MHD descriptions of the disk velocities and particle locations. The dust over densities tend to be much larger and take longer to dissipate.

In order to be unstable to collapse, two criteria must be satisfied. The clump of solid material must have a radius smaller than the Jeans radius;

$$R_J = \frac{2GM}{\sigma^2}$$

where σ is the velocity dispersion of the clump. The lifetime of the clump must also be long enough that the free-fall collapse is allowed to take place.

$$t_{ff} = \sqrt{\frac{R^3}{GM}} < t_{cl}.$$

These models also have their limitations. These simulations used magneto-rotational turbulence. This may not be strictly appropriate at the midplane, where ionization due to radiation and cosmic particles is low. When considering stability to collapse, it is assumed that half of the disk mass is present in the boulders modelled, but in fact much more mass may be present in smaller dust grains. No back-reaction from the dust on the gas is included, and any self-gravity of the disk is ignored.

References and further reading

For further reading and more details on the topics covered, please see the following works;

Overview with emphasis on Planetesimal hypothesis

Lissaeur J.J., 1993, *Annual Review of Astronomy and Astrophysics*, 31:129-174.

Motions of solids in turbulent disk

Völk, H.J., et al., 1980, *Astronomy and Astrophysics*, 85:316-325

Mizuno H., et al., 1988, *Astronomy and Astrophysics*, 195:183-192 *and references therein*

Calculating opacity in complex media

Ossenkopf V., 1991, *Astronomy and Astrophysics*, 251:210

Ossenkopf V., 1993, *Astronomy and Astrophysics*, 280:617

Radial one-dimensional disk modelling

Schmitt W., Henning T., Mucha R., 1997, *Astronomy and Astrophysics*, 325:569-584

Vertial one-dimensional disk modelling

Weidenschilling S.J., 1984, *Icarus*, 60:553-567

Monte-Carlo techniques of later stages of planet formation

Wetherill G.W., 1990, *Annual Review of Earth and Planetary Sciences*, 18:205-256 *and references therein*

Gas accretion

Pollack J.B., et al., 1996, *Icarus* 124:62-85

Gravitational formation of planets in non-turbulent disks

Goldreich P., Ward W.R., 1973, *Astrophysical Journal*, 183:1051.

New gravo-turbulent collapse models

Johansen A., Klahr H., Henning T., 2006, *Astrophysical Journal*, 636:1121-1134

And look out for the conference proceedings Protostars and Planets V, due to be published soon, for reviews of recent work of all areas of star and planet formation.

Appendix A : Van der Waals forces

Van der Waals forces are forces which exist between molecules of the same substance. They are of three basic types;

Dipole - Dipole forces;

These occur in polar molecules, i.e. where there is an unequal sharing of electrons in the molecule. Hydrogen chloride is such a molecule, the chlorine has an extra electron from the hydrogen, and so the chlorine part of the molecule is negatively charged, and the hydrogen part is positively charged. In solution, where many molecules have a positive and negative side, the molecules will naturally orient themselves so that a positive side is next to a negative side of a neighbouring particle. The attractive forces will then 'stick' the molecules together.

Dispersion forces;

These forces exist between non-polar molecules. There is no overall charge on either side of the molecule. However, electrons are constantly in motion, and so, on very small but finite timescales, a temporary imbalance in which the electrons are more dominant on one side may arise, creating a temporary dipole. These temporary dipoles allow a temporarily negative side to attract a temporarily positive side of another molecule, resulting in a intermolecular force.

Hydrogen bonding;

This is exactly the same as the dipole-dipole interaction, but is given a special name. It describes the dipole-dipole interaction between any molecule with a bond between hydrogen and oxygen, fluorine or nitrogen. This is in part due to the unusual strength of the hydrogen bonding in say water or ammonia, compared to other van der Waals dipole-dipole forces, and also due to its importance in, say, the relatively high boiling point of water. The bond is so strong because oxygen, fluorine and nitrogen are extremely good at attracting electrons, and hydrogen is very good at losing them.Thus the negative charge around the electron attracting atom is highly concentrated, and thus the attraction between this and a positive hydrogen from a neighbouring molecule is very strong.

Appendix B : MHD equations in Johansen et al.

The shearing sheet approximation with an additional pressure gradient of $P \propto r^{\alpha}$ through the box produces the following (including variable transformation $\ln\rho \to \ln\rho + (1/r_0)\alpha x$),

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\Delta)\mathbf{u} = -2\Omega_0 \times \mathbf{u} + 3\Omega_0^2 \mathbf{x} - c_s^2 \Delta \ln\rho - c_s^2 \frac{1}{r_0}\alpha\hat{\mathbf{x}}.$$

The first term on the right is the Coriolis force, the second is the combined effects of gravitation in the radial direction and centrifugal acceleration, and the final terms are the effects of the local and global pressure gradient. It it the final term that comes from taking the limit in $\ln\rho \to \ln\rho + (1/r_0)\alpha x$.

$\mathbf{x}$ is measured from the co-moving radial position $\mathbf{r}_0$ from the central source of gravity, the x direction pointing radially outwards, the y direction along the Keplerian flow.

Balance between centrifugal force, the pressure gradient and gravity for the sub-Keplerian rotation of the disk gives

$$u_y^{(0)} = -\frac{3}{2}\Omega_0 x + \frac{c_s^2}{2\Omega_0}\frac{1}{r_0}\alpha,$$

where the first right-hand term is simply the Keplerian rotation profile and the second term is constant, the adjustment due to the global pressure gradient. Now if we measure all velocities relative to this sub-Keplerian flow, we make the transformation $\mathbf{u} \to \mathbf{u} + \mathbf{u}_0$ where $\mathbf{u}_0 = u_y^{(0)}\hat{\mathbf{y}}$. Thus our gas motions are described by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\Delta)\mathbf{u} + u_y^{(0)}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{f}(\mathbf{u}) - c_s^2 \Delta \ln\rho$$

where the function

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 2\Omega_0 \\ -\frac{1}{2}\Omega_0 u_x \\ 0 \end{pmatrix}.$$

Similarly for the equation of motion of the dust,

$$\frac{\partial \mathbf{v}_i}{\partial t} = \mathbf{f}(\mathbf{v_i}) - \frac{1}{\tau_f}(\mathbf{v_j} - \mathbf{u}) + c_s^2\frac{1}{r_0}\alpha\hat{\mathbf{x}}.$$

Here there is no pressure term to balance the extra coriolis force, and so the final term is the head wind experienced by the particle. Magnetohydrodynamic equations are now used to describe the gas velocity, magnetic vector potential $\mathbf{A}$, gas denstiy ρ, particle velocities $\mathbf{v}_i$ and particle coordinates $\mathbf{x}_i$,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\Delta)\mathbf{u} + u_y^{(0)}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{f}(\mathbf{u}) - c_s^2 \Delta \ln\rho + \frac{1}{\rho}\mathbf{J}\times\mathbf{B} + \mathbf{f}_\nu(\mathbf{u},\rho),$$

$$\frac{\partial \mathbf{A}}{\partial t} + u_y^{(0)}\frac{\partial \mathbf{A}}{\partial y} = \mathbf{u}\times\mathbf{B} + \frac{3}{2}\Omega_0 A_y \hat{\mathbf{x}} + \mathbf{f}_\eta(\mathbf{A}),$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\Delta\rho + u_y^{(0)}\frac{\partial \rho}{\partial y} = -\rho\Delta\cdot\mathbf{U} + f_D,$$

$$\frac{\partial \mathbf{v}_i}{\partial t} = \mathbf{f}(\mathbf{v_i}) - \frac{1}{\tau_f}(\mathbf{v}_i - \mathbf{u}) + c_s\Omega_0\beta\hat{\mathbf{x}},$$

$$\frac{\partial \mathbf{x}_i}{\partial t} = \mathbf{v}_i + u_y^{(0)}\hat{\mathbf{y}}.$$

$\beta = \alpha H/r_0$ where $H = c_s/\Omega_0$ is the scale height of the disk, we call β the pressure gradient parameter. $\mathbf{f}_\nu, \mathbf{f}_\eta$ and f_D are hyperdiffusity terms added here to stabilise the finite difference numerical scheme of the code used to model the disk.