Structure Formation in the Linear Regime

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1 Introduction

We believe that the large-scale Universe should be described by general relativity and the Cosmological Principle, which states:

Cosmological Principle: On sufficiently large scales, the Universe is homogeneous and isotropic.

Homogeneity means constant density and *isotropy* means the same in all directions, so the cosmological principle tells us that there is no preferred direction or position in the Universe.
From this starting point we can find the metric which describes the large scale Universe. The requirements of the cosmological principle allow only the *Robertson-Walker metric*, from which we can obtain the Einstein tensor $G^{\mu\nu}$. General relativity tells us that the Universe should satisfy the Einstein equations:

$$G^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu} \tag{1}$$

and the Cosmological Principle tells us that we should use the perfect fluid form of the energy-momentum (or stress-energy) tensor $T^{\mu\nu}$. Now the Einstein equations lead to the Friedmann equations describing the evolution of such a Universe.

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3}$$

where H is the Hubble parameter, ρ is the energy density and a is the scale factor.
However, when we observe the Universe, we see structures such as galaxies, clusters and superclusters so there is obviously a scale at which the Cosmological Principle ceases to hold. If this were not the case, there would be no structure and we would not exist. We want to describe the Universe that we observe, and the usual approach is to model this as a perturbation to a smooth background which we assume is homogeneous and isotropic. This perturbed Universe should the evolve under gravity to that which we observe today. We often assume an inflationary model for the very early Universe where quantum fluctuations provide the seeds for these small perturbations. The simplest models of inflation predict that these small perturbations about the background

will be Gaussian and a lot of work is being carried out at present to try to detect non-Gaussianity in the initial perturbations which would rule such models out.

The CMB is a rich source of information about these perturbations in the early Universe. The very fact that we observe small fluctuations in the otherwise uniform background of the CMB is evidence that such perturbations exist and the size of the temperature inhomogeneities constrains the amplitude of the density perturbations at the epoch of last scattering.

The remainder of this seminar will concentrate on the evolution of these small perturbations to the smooth background Universe, developing the ideas behind structure formation in the linear regime.

2 Dynamics of structure formation

A full treatment of these perturbations would require general relativity and would proceed by perturbing the background metric and the energy-momentum tensor, see (Padmanabhan, 1993). However, we consider here a Newtonian approximation rather than the full general relativistic treatment. This is sufficient for small enough perturbations because small perturbations imply that we are considering weak fields and in such cases the Newtonian approximation is valid. In general, this analysis holds for perturbations on scales much smaller than the Hubble radius c/H. Perturbations on larger scales require a full general relativistic treatment to deal with gauge issues. The results that we would obtain from a full GR treatment are formally similar to those obtained from a Newtonian approximation (on scales where this is valid) but far messier to derive.

We will work in a proper coordinate system with position given by $\vec{\boldsymbol{x}}$ and velocity $\vec{\boldsymbol{v}}$. In this coordinate system, the homogeneous background has velocities given by the Hubble law $\vec{\boldsymbol{v}}_0 = H\vec{\boldsymbol{x}}$. All smooth background quantities are labelled with the subscript 0 and small perturbations to the background are $\delta\rho, \delta p$ etc. Growth of inhomogeneities in the density field can begin as soon as the Universe becomes matter dominated (although baryonic inhomogeneities cannot grow until after decoupling). We consider below a single fluid model for collisional matter since it is then an easy matter of ignoring the pressure terms to convert this to a model for collisionless matter.

2.1 Equations of fluid motion

We begin with the equations of motion for a non-relativistic fluid (since the Cosmological Principle tells us that the smooth background Universe can be described as a perfect fluid):

$$\frac{D\rho}{Dt} = -\rho\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{v}} \qquad \text{Continuity Equation} \tag{4}$$

$$\frac{D\vec{\boldsymbol{v}}}{Dt} = -\frac{\vec{\boldsymbol{\nabla}}\boldsymbol{p}}{\rho} - \vec{\boldsymbol{\nabla}}\boldsymbol{\Phi} \qquad \text{Euler Equation} \tag{5}$$

$$\nabla^2\Phi = 4\pi G\rho \qquad \text{Poisson equation} \tag{6}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}}) \tag{7}$$

is the *convective derivative* i.e. the derivative with respect to the fluid motion and ρ is the density, $\vec{\boldsymbol{v}}$ the velocity, p the pressure and Φ the potential field.

- The continuity equation is simply an expression of conservation of mass. It states that the change in the mass inside an element of the fluid is equal to the mass convected into the element - the mass convected out of the element.
- The Euler equation states that the acceleration of a small fluid element (per unit volume) $\rho D\vec{\boldsymbol{v}}/Dt$, is due to the difference in pressure acting on opposite sides of the element (per unit volume) $-\vec{\boldsymbol{\nabla}}\boldsymbol{p}$ and the gravitational force (per unit volume) acting on the element $-\rho\vec{\boldsymbol{\nabla}}\boldsymbol{\Phi}$.
- The Poisson equation tells us about the relation between the potential fluctuations and the density perturbations which cause them.

These equations hold for the smooth background i.e.:

$$\frac{\partial\rho_0}{\partial t} + (\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})\rho_0 = -\rho_0\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{v}}_0 \tag{8}$$

$$\frac{\partial\vec{\boldsymbol{v}}_0}{\partial t} + (\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0 = -\frac{\vec{\boldsymbol{\nabla}}\boldsymbol{p_0}}{\rho_0} - \vec{\boldsymbol{\nabla}}\boldsymbol{\Phi_0} \tag{9}$$

$$\nabla^2\Phi_0 = 4\pi G\rho_0 \tag{10}$$

where (as $\vec{\boldsymbol{v}}_0 = H\vec{\boldsymbol{x}}$ and ρ_0 is homogeneous) they simplify to give:

$$\frac{\partial\rho_0}{\partial t} = -3H\rho_0 \tag{11}$$

$$\frac{\partial\vec{\boldsymbol{v}}_0}{\partial t} + H\vec{\boldsymbol{v}}_0 = -\frac{\vec{\boldsymbol{\nabla}}\boldsymbol{p_0}}{\rho_0} - \vec{\boldsymbol{\nabla}}\boldsymbol{\Phi_0} \tag{12}$$

$$\nabla^2\Phi_0 = 4\pi G\rho_0 \tag{13}$$

2.2 First order solutions

Consider small perturbations to the background density, velocity, pressure and potential fields i.e. $\delta\rho/\rho_0 \ll 1$ etc:

$$\rho = \rho_0 + \delta\rho$$

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_0 + \delta\vec{\boldsymbol{v}}$$

$$p = p_0 + \delta p$$

$$\Phi = \Phi_0 + \delta\Phi$$

Substituting the above into equations (4), (5) and (6) and ignoring terms higher than first order in the perturbations gives:

$$
\begin{aligned}
\frac{\partial\rho_0}{\partial t}+\frac{\partial\delta\rho}{\partial t}+(\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})\rho_0+(\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})\delta\rho+(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\rho_0 &= -\rho_0\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{v}}_0-\rho_0\vec{\boldsymbol{\nabla}}.\delta\vec{\boldsymbol{v}}-\delta\rho\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{v}}_0\\
\frac{\partial\vec{\boldsymbol{v}}_{\mathbf{0}}}{\partial t}+\frac{\partial\delta\vec{\boldsymbol{v}}}{\partial t}+(\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0+(\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})\delta\vec{\boldsymbol{v}}+(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0 &= -\frac{\boldsymbol{\nabla p_0}\vec{+}\boldsymbol{\nabla\delta p}}{\rho_0+\delta\rho}-\boldsymbol{\nabla}\vec{\boldsymbol{\Phi}}_{\mathbf{0}}-\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{\Phi}\\
\nabla^2\Phi_0+\nabla^2\delta\Phi &= 4\pi G\rho_0+4\pi G\delta\rho
\end{aligned}
$$

Now we can subtract the zero order equations (8), (9) and (10) from the above perturbed equations and if we set $\frac{d}{dt}=\frac{\partial}{\partial t}+(\vec{\boldsymbol{v}}_0.\vec{\boldsymbol{\nabla}})$ then we obtain:

$$
\begin{aligned}
\frac{d\delta\rho}{dt}+(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\rho_0 &= -\rho_0\vec{\boldsymbol{\nabla}}.\delta\vec{\boldsymbol{v}}-\delta\rho\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{v}}_0\\
\frac{d\delta\vec{\boldsymbol{v}}}{dt}+(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0 &= \frac{\boldsymbol{\nabla}\vec{\boldsymbol{p}}_{\mathbf{0}}}{\rho_0}\delta-\frac{\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{p}}{\rho_0}-\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{\Phi}\\
\nabla^2\delta\Phi &= 4\pi G\delta\rho
\end{aligned}
$$

Now recall the homogeneity of ρ_0 and assume an equation of state for the smooth background $p_0=w\rho_0$ where w is independent of position (this is a condition of adiabatic perturbations) to find:

$$
\begin{aligned}
\frac{d\delta}{dt} &= -\vec{\boldsymbol{\nabla}}.\delta\vec{\boldsymbol{v}}\\
\frac{d\delta\vec{\boldsymbol{v}}}{dt}+(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0 &= -\frac{\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{p}}{\rho_0}-\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{\Phi}\\
\nabla^2\delta\Phi &= 4\pi G\delta\rho
\end{aligned}
$$

where $\delta=\delta\rho/\rho_0$.
Considering the components of $(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0$ we see (Einstein summation convention):

$$[(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0]_k=\delta v_j\nabla_j(v_0)_k$$

But $\vec{\boldsymbol{v}}_0=H\vec{\boldsymbol{x}}$ where H is the Hubble constant. So,

$$[(\delta\vec{\boldsymbol{v}}.\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}}_0]_k=\delta v_j\nabla_jHx_k=H\delta v_j\delta^j_k=H\delta v_k \tag{14}$$

Hence, the first order perturbation equations are:

$$
\boxed{
\begin{aligned}
\frac{d\delta}{dt} &= -\vec{\boldsymbol{\nabla}}.\delta\vec{\boldsymbol{v}} &(15)\\
\frac{d\delta\vec{\boldsymbol{v}}}{dt} &= -H\delta\vec{\boldsymbol{v}}-\frac{\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{p}}{\rho_0}-\boldsymbol{\nabla}\vec{\boldsymbol{\delta}}\boldsymbol{\Phi} &(16)\\
\nabla^2\delta\Phi &= 4\pi G\rho_0\delta &(17)
\end{aligned}
}
$$

We now have a set of 3 equations for 4 unknowns so one final equation is needed to complete the set. This is the equation relating density and pressure:

$$c_s^2=\frac{\delta p}{\delta\rho} \tag{18}$$

where c_s is the sound speed.

These equations hold as long as the perturbations remain small. Once they become large, linear theory fails and the above becomes invalid. For non-linear scales we need another approach to investigate structure formation. In general the full non-linear problem needs to be analysed numerically and this is where N-body simulations come into their own. See Barney's seminar next week for further details of the non-linear regime.

3 Comoving coordinates

The above equations have been formulated in *Eulerian* coordinates, using proper lengths. Often in fluid dynamics we choose to use *Lagrangian* coordinates which move with the fluid. This means that the Lagrangian position of a particle in the fluid is fixed in time.

As with many situations in cosmology the analysis above becomes simpler if we switch to a coordinate system that incorporates the background expansion, these are *comoving* coordinates. In a smooth Universe, comoving coordinates are an example of a Lagrangian coordinate system. However, when we consider a perturbed version of this Universe comoving coordinates are no longer Lagrangian as they do not completely move with the fluid. Although the background expansion is incorporated, peculiar velocities (components of the velocity of an object *not* due to the background expansion) are not.

To put the above equations into comoving coordinates we need the position coordinates to change to incorporate the expansion factor given by $a(t)$:

$$\vec{\boldsymbol{x}} = a\vec{\boldsymbol{r}} \tag{19}$$

where $\vec{\boldsymbol{x}}$ is the Eulerian position and $\vec{\boldsymbol{r}}$ is the comoving position. Similarly, the peculiar velocity will change to:

$$\delta\vec{\boldsymbol{v}} = a\vec{\boldsymbol{u}} \tag{20}$$

where $\vec{\boldsymbol{u}}$ is the comoving peculiar velocity. Now:

$$\vec{\boldsymbol{\nabla}}_p = \frac{\partial}{\partial\vec{\boldsymbol{x}}} = \frac{\partial}{a\partial\vec{\boldsymbol{r}}} = \frac{\vec{\boldsymbol{\nabla}}_c}{a} \tag{21}$$

where the p and c subscripts refer to proper (Eulerian) and comoving coordinates respectively. If $\vec{\boldsymbol{g}} = -\vec{\boldsymbol{\nabla}}_p\delta\Phi$ then rewriting equations (15), (16) and (17) in comoving coordinates gives:

$$\boxed{\begin{aligned} \dot{\delta} &= -\vec{\boldsymbol{\nabla}}_c.\vec{\boldsymbol{u}} && (22) \\ \dot{\vec{\boldsymbol{u}}} + 2H\vec{\boldsymbol{u}} &= \frac{\vec{\boldsymbol{g}}}{a} - \frac{\vec{\boldsymbol{\nabla}_c\delta p}}{a^2\rho_0} && (23) \\ \nabla_c^2\delta\Phi &= 4\pi G\rho_0\delta a^2 && (24) \end{aligned}}$$

where $\dot{} = d/dt$.

4 Growing modes

The above equations tell us about the evolution of the density, velocity, potential and pressure fields. We are interested in the formation of structures and therefore we wish to know in more detail about the evolution of the density field. Beginning with the continuity equation we can eliminate the velocity field using the Euler equation to obtain a second order differential equation for the density field.

$$\begin{aligned}
\dot{\delta} &= -\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{u}} \\
\Rightarrow \ddot{\delta} &= -\vec{\boldsymbol{\nabla}}.\dot{\vec{\boldsymbol{u}}} = -\vec{\boldsymbol{\nabla}}.\left(-2H\vec{\boldsymbol{u}} + \frac{\vec{\boldsymbol{g}}}{a} - \frac{\nabla^2 \delta p}{a^2 \rho_0}\right) \\
\Rightarrow \ddot{\delta} + 2H\dot{\delta} &= 4\pi G \rho_0 \delta + \frac{c_s^2 \nabla^2 \delta}{a^2}
\end{aligned} \tag{25}$$

This equation is often known as the *growth equation.* If we allowed several different components to be present in the fluid that we are modelling then the evolution of a perturbation δ_i in a *non-relativistic* component i is:

$$\ddot{\delta}_i + 2H\dot{\delta}_i = 4\pi G \rho_0 \sum_j \frac{\rho_j}{\rho_0} \delta_i + \frac{c_s^2 \nabla^2 \delta_i}{a^2} \tag{26}$$

Considering a single component model again, move to Fourier space, expanding the density perturbation in Fourier modes $\delta \propto e^{-i\vec{\boldsymbol{k}}.\vec{\boldsymbol{r}}}$. Each k mode evolves independently and hence:

$$\begin{aligned}
\ddot{\delta}_k + 2H\dot{\delta}_k &= 4\pi G \rho_0 \delta_k - \frac{c_s^2 k^2 \delta_k}{a^2} \\
\Rightarrow \ddot{\delta}_k + 2H\dot{\delta}_k &= \left(4\pi G \rho_0 - \frac{c_s^2 k^2}{a^2}\right) \delta_k
\end{aligned} \tag{27}$$

The sign of the term in brackets on the right hand side of equation (27) determines the behaviour of the density perturbations. In the absence of the damping term $2H\dot{\delta}$ (equivalent to considering this equation in a non-expanding Universe), if the term in brackets were positive $4\pi G\rho_0 > c_s^2 k^2/a^2$ then we would have growing and decaying solutions given by the exponentials e^{At}, e^{-At} where $A = \sqrt{4\pi G\rho_0 - c_s^2 k^2/a^2}$. If the term in brackets were negative we would have oscillatory solutions. The damping term makes such an analysis more complex but it remains true that when $4\pi G\rho_0 \gg c_s^2 k^2/a^2$ we will have growing and decaying solutions (although due to the damping, power law solutions are more likely than exponentials) and when $4\pi G\rho_0 \ll c_s^2 k^2/a^2$ we will have oscillatory solutions.

Physically, this different behaviour is due to whether or not pressure in the fluid can be neglected. If this is the case we have growing and decaying modes and if not we have oscillatory solutions because gravity acts to make structures collapse and pressure acts to oppose it which sets up an oscillation. The physical scale at which the gravitational ($4\pi G\rho_0$) and pressure ($c_s^2 k^2/a^2$) terms balance is called the *Jeans length*, $\lambda_J = c_s\sqrt{\pi/G\rho}$. Long wavelength modes ($\lambda \gg \lambda_J$) can grow and short wavelength modes ($\lambda \ll \lambda_J$) oscillate. As λ_J is time dependent in an expanding Universe, a given mode λ can have periods of growth and oscillation governed by the evolution of λ_J

For collisionless matter, there is no pressure term and the equation for the growth of density perturbations simplifies to:

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_0\delta \tag{28}$$

This equation can be solved analytically in certain special cases, otherwise it must be solved numerically or a numerical approximation to the solution must be employed.

In the case of a Universe containing dark matter and dark energy with a constant equation of state w we can see that the Hubble constant is a solution to equation (28) for certain values of w:

$$\begin{aligned}
H^2 &= \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{kc^2}{a^2} \\
\Rightarrow 2H\dot{H} &= \frac{8\pi G}{3}(\dot{\rho}_m + \dot{\rho}_\Lambda) + \frac{2kc^2\dot{a}}{a^3} \\
\Rightarrow 2H\dot{H} &= -8\pi GH(\rho_m + \rho_\Lambda(1+w)) + \frac{2kc^2\dot{a}}{a^3} \text{ using } \dot{\rho} = -3H\rho(1+w) \\
\Rightarrow \dot{H} &= -4\pi G(\rho_m + \rho_\Lambda(1+w)) + \frac{kc^2}{a^2} \\
\Rightarrow \ddot{H} &= 12\pi GH(\rho_m + \rho_\Lambda(1+w)^2) - \frac{2kc^2\dot{a}}{a^3} \\
\text{Hence,} & \\
\ddot{H} + 2H\dot{H} &= 12\pi GH(\rho_m + \rho_\Lambda(1+w)^2) + -8\pi GH(\rho_m + \rho_\Lambda(1+w)) - \frac{2kc^2H}{a^2} + \frac{2kc^2H}{a^2} \\
&= 4\pi GH(\rho_m + \rho_\Lambda\left(-2(1+w) + 3(1+w)^2\right))
\end{aligned}$$

Hence, $\delta = H$ solves equation (28) if $1 + 4w + 3w^2 = 0$ or if $\rho_\Lambda = 0$ i.e. if $w = -1/3$ or $w = -1$ or $\rho_\Lambda = 0$.

In these cases (assuming now a flat Universe) H is a decaying mode as we can see from $\dot{H} = -4\pi G(\rho_m + \rho_\Lambda(1+w)) \leq 0$ with $\rho_m, \rho_\Lambda \geq 0$ and $w = -1/3, -1$. In the case of structure formation, the growing mode is of most interest since, as its name suggests, the decaying mode decreases to become negligible over time. This decaying mode corresponds to a situation where a perturbation with initial overdensity and velocity are arranged such that the initial velocity perturbation eventually 'undoes' the density perturbation. To find the growing mode in any of the cases described above ($w = -1, w = -1/3$ etc), we use the reduction of order method.

Assume that the second solution has the form $\delta_2(t) = f(t)\delta_1(t) = f(t)H(t)$. Then:

$$\begin{aligned}
\dot{\delta_2} &= \dot{f}\delta_1 + f\dot{\delta}_1 \\
\ddot{\delta_2} &= \ddot{f}\delta_1 + 2\dot{f}\dot{\delta}_1 + f\ddot{\delta}_1 \\
\Rightarrow \ddot{\delta}_2 + 2H\dot{\delta}_2 + 4\pi G\rho_0\delta_2 &= \ddot{f}\delta_1 + 2\dot{f}\dot{\delta}_1 + f\ddot{\delta}_1 + 2H\left(\dot{f}\delta_1 + f\dot{\delta}_1\right) + 4\pi G\rho_0 f\delta_1 \\
\Rightarrow \ddot{f}\delta_1 + 2\dot{f}\dot{\delta}_1 + 2H\dot{f}\delta_1 &= 0 \\
\frac{\ddot{f}}{\dot{f}} &= -\frac{2\dot{\delta}_1 + 2H\delta_1}{\delta_1} \\
\left[\log\dot{f}\right] &= -\left[2\log\delta_1 + 2\log a\right] \text{ where a is the scale factor} \\
\Rightarrow f(t) &= B\int\frac{1}{\delta_1^2 a^2}dt = B\int\frac{1}{\dot{a}^2}dt \text{ where B is a constant}
\end{aligned} \tag{29}$$

Usually the value of the constant B is found by requiring that $\delta_2 = a$ in the matter-dominated era, then the growing mode solution is:

$$\delta_2 = \frac{5}{2} H_0^2 \Omega_{m,0} H \int \frac{1}{\dot{a}^2} dt \tag{30}$$

A popular numerical approximation for the growing mode solution to equation (28) in a flat Universe with a cosmological constant is (Carroll et al., 1992):

$$\delta_2 = \frac{5}{2} \frac{\Omega_m a}{\Omega_m^{4/7} - \Omega_\Lambda + \left(1 + \frac{\Omega_\Lambda}{70}\right)\left(1 + \frac{\Omega_m}{2}\right)} \tag{31}$$

More recently (Percival, 2005), a fit to the growing mode in a flat Universe with dark energy which has a constant equation of state has been found.

5 Uses

What use is all of this? The above analysis applies in the linear regime for perturbations on scales much smaller than the horizon, but what exactly does this mean?
In the early Universe, when we assume that the perturbations were small, everything was linear and hence this linear analysis can be used for early times. As we evolve to approach the present day, structures begin to form by gravitational instability and linear theory ceases to be a valid approximation on increasingly larger scales. Perturbations become non-linear ($\delta\rho/\rho \sim 1$) on small scales first because the power is greater here and as time progresses, modes become non-linear on increasingly larger scales. Today, modes with $k \lesssim 0.2\, h\mathrm{Mpc}^{-1}$ are described to reasonable accuracy by linear theory but larger k-modes (which correspond to smaller physical scales) are not.
This may sound quite restrictive, but in fact due to the complications that arise when things become non-linear, linear methods are often a starting point for more involved analysis, even on small scales. On large scales, the results of linear theory have many applications. For example, they allow us to compare our models of the Universe with observations to constrain cosmological parameters and to infer properties of the power spectrum of density fluctuations. Also, the growth equation and in particular its growing mode solution are sensitive to the precise nature of the dark energy which we believe currently dominates the Universe. Some of the upcoming CMB and supernova studies (for example Planck and SNAP) should be able to make use of the growing mode to constrain dark energy parameters (Cooray et al., 2004).

5.1 Velocity fields (not covered today)

Inhomogeneities in the matter distribution imply the existence of small fluctuations in the CMB as mentioned above, but they also imply that particles should have peculiar velocities i.e. velocities that are not due to the Hubble flow. The comoving peculiar velocity field $\vec{u}$ is related to the density field δ in the manner given by equations (22) and (23). Moving to Fourier space allows us to re-express the velocity field in terms of the density field.

Before we begin, it is useful to decompose the velocity field into rotational and irrotational terms $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_{rot} + \vec{\boldsymbol{u}}_{irrot}$ where $\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{u}}_{rot} = 0$ and $\vec{\boldsymbol{\nabla}} \wedge \vec{\boldsymbol{u}}_{irrot} = 0$ so $\vec{\boldsymbol{k}}.\vec{\boldsymbol{u}}_{rot} = 0$ and $\vec{\boldsymbol{k}}.\vec{\boldsymbol{u}}_{irrot} = |\vec{\boldsymbol{k}}||\vec{\boldsymbol{u}}_{irrot}|$. Now from equations (22) and (23) we find:

$$\dot{\delta}_k - \frac{i\vec{\boldsymbol{k}}.\vec{\boldsymbol{u}}_{irrot}}{a} = \dot{\delta}_k - \frac{iku}{a} = 0 \tag{32}$$

where $k = |\vec{\boldsymbol{k}}|$ and $u = |\vec{\boldsymbol{u}}_{irrot}|$. Then

$$u = \frac{a\dot{\delta}_k}{ik} \Rightarrow \vec{\boldsymbol{u}}_{irrot} = \frac{a\dot{\delta}_k}{ik}\hat{\vec{\boldsymbol{k}}} \tag{33}$$

From the Euler equation (23),

$$\frac{d(a\vec{\boldsymbol{u}}_{rot} + a\vec{\boldsymbol{u}}_{irrot})}{dt} - i\vec{\boldsymbol{k}}c_s^2\delta_k - i\vec{\boldsymbol{k}}\delta\Phi = 0 \Rightarrow \frac{d\vec{\boldsymbol{u}}_{rot}}{dt} = 0 \Rightarrow \vec{\boldsymbol{u}}_{rot} = \frac{const.}{a} \tag{34}$$

Hence we see that $\vec{\boldsymbol{u}}_{rot}$ decays and hence if we consider a late enough epoch, this mode will be negligible. The peculiar velocity field can therefore be expressed entirely in terms of the irrotational mode which is proportional to $\vec{\boldsymbol{k}}$. We therefore drop the *irrot* subscript in the following. From the continuity equation (22) once again:

$$\begin{aligned} \dot{\delta}_k &= \frac{i}{a}\vec{\boldsymbol{k}}.\vec{\boldsymbol{u}}_k \\ \vec{\boldsymbol{u}}_k &= -\frac{ia}{k}\hat{\vec{\boldsymbol{k}}}\dot{\delta}_k \end{aligned}$$

Recalling the expression for the growing mode given above, we can write:

$$\dot{\delta} = \dot{a}\frac{d\delta}{da} = H\delta f(\Omega) \text{ where } f = \frac{a}{\delta}\frac{d\delta}{da} \tag{35}$$

Hence:

$$\begin{aligned} \vec{\boldsymbol{u}}_k &= -\frac{ia}{k^2}Hf\vec{\boldsymbol{k}}\delta_k \\ \frac{-k^2}{a^2}\delta\Phi_k &= \frac{i}{a}\vec{\boldsymbol{k}}.\vec{\boldsymbol{g}}_k = 4\pi G\rho_0\delta_k \text{ (Poisson)} \end{aligned}$$

but by definition $\vec{\boldsymbol{g}}_k$ is parallel to $\vec{\boldsymbol{k}}$ so:

$$\begin{aligned} \Rightarrow \vec{\boldsymbol{u}}_k &= \frac{Hf}{4\pi G\rho_0}\vec{\boldsymbol{g}}_k \\ \Rightarrow \vec{\boldsymbol{u}}_k &= \frac{2f}{3H\Omega}\vec{\boldsymbol{g}}_k \end{aligned} \tag{36}$$

Hence in real space,

$$\boxed{\vec{\boldsymbol{u}} = \frac{2f}{3H\Omega}\vec{\boldsymbol{g}}} \tag{37}$$

Linear perturbation theory therefore implies that peculiar velocities are proportional to gravitational acceleration.

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