# Deriving the Stellar Initial Mass Function from Turbulent Fragmentation

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## 1 Introduction

The Initial Mass Function (IMF) is the most important diagnostic tool for models of star formation. Any such models must produce an IMF consistent with observation.

## 2 The IMF

The IMF is defined as the number of stars formed in a unit volume in a mass range $dlog_{10}m$.

$$\xi(log_{10}m) = \frac{dn}{dlog_{10}m} \tag{1}$$

The landmark study of the IMF was Salpeter (1955) [10]. Here the IMF was found to have a power law form.

$$\xi(log_{10}m) \propto m^{-x} \tag{2}$$

However more recent studies have shown that at lower masses the IMF flattens off and takes a lognormal form such that,

$$log_{10}\xi(log_{10}m) = a_0 + a_1 log_{10}m + a_2(log_{10}m)^2 \tag{3}$$

which can also be written as,

$$\xi(log_{10}m) \propto exp\left(\frac{(log_{10}m - log_{10}m_c)^2}{2\sigma^2}\right) \tag{4}$$

where $\sigma$ is the variance and $log_{10}m_c$ is some characteristic mass.

### 2.1 Studies of the IMF

The defining paper in the observation of the stellar IMF came with Salpeter (1955) [10]. Here the mass function was found to be a power law with $x = 1.35$ for stars of a few solar masses and above. However the IMF has been shown to flatten off at lower masses by Miller & Scalo (1978) [6] and take a lognormal form. Most recently Chabrier (2003) [4] found that $log_{10}m_c = -1.1$ and $\sigma = 0.79$. The IMF is not found to be discontinuous across the hydrogen burning limit $M = 0.075M_\odot$. The high mass IMF has also been found to be the same in different environments. Baldry & Glazebrook [2] used population synthesis techniques for galaxies at $z = 0.1$ to find that the IMF was well fitted by a power law with an index of $1.15 \pm 0.2$, consistent with Salpeter. Veltchev [11] finds that the high mass IMF in M31 is a power law of slope $x = 1.59 \pm 0.09$, again similar (but not identical) to the Salpeter slope.

### 2.2 Models of Star Formation

The basic picture of star formation is the following. A molecular cloud fragments due to turbulence forming dense prestellar cores. Prestellar cores have been found to have a power law mass function with $x = 1.5$ Motte, Andre & Neri (1998) [7]. How these cores collapse is still an open question. Adams & Fatuzzo [1] claim that the form of the IMF is somehow related to the central limit theorem, with many processes in the molecular cloud leading to a lognormal form. However this relies on an almost infinite number of physical processes. Also one suggested process involves mass loss due to stellar winds. Clearly this would imply a discontinuity at the H-burning limit; one which does not exist.
Bate & Bonnell [3] state that all stars begin life as objects of about $0.003M_\odot$ and increase in mass by a process of competetive accretion. Here prestellar cores do not form individual stars but several stars. This process is based on the results of numerical simulations rather than having an analytical basis.
The only analytical model to explain both the power law form at high masses and the lognormal form at low masses is that of superalfvenic turbulence found in Padoan & Nordlund [8].

## 3 Basic Concepts

### 3.1 The Reynolds Number and Turbulence

The Reynolds Number of a fluid is the ratio of the inertial force to the viscous force.

$$Re = \frac{\rho v_L L}{\mu} \tag{5}$$

where $\mu$ is the viscosity, L is the scale the flow is being examined on and $v_L$ is the velocity associated with that scale. At low Re the viscous force dominates and the flow is laminar and can be described analytically. However at high Re the viscous force becomes smaller and the flow becomes turbulent. Turbulent flows contain random components to the velocity and hence are only describable statistically.

### 3.2 Turbulent Energy Cascade

We can imagine turbulence as a series of eddies of a scale $L$ with energy injected at a large scale (say by the galactic wind) and cascading down to the smallest scale eddies where the energy is dissipated. It can be shown that the velocity $v_L$ at a certain scale $L$ can be shown to obey the following relation.

$$v_L \propto L^\alpha \tag{6}$$

This has been observed in molecular clouds by Larson (1981) [5] and is known as the Larson Relation. Observational studies have pinned the value of $\alpha$ down to 0.4. The Energy Power spectrum of the turbulence at a particular wavenumber $k$ is,

$$E(k) \propto k^{-\beta} \tag{7}$$

where

$$\alpha = \frac{\beta - 1}{2} \tag{8}$$

### 3.3 Jeans Mass

The mass at which a cloud cannot support itself from gravitational collapse by thermal pressure is the Jeans Mass.

$$m_J = \left(\frac{5kT}{\mu G m_H}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} \tag{9}$$

Here $\mu$ is the mean molecular mass. This can also be written as,

$$m_J = m_{J,0} \left(\frac{n}{n_0}\right)^{-1/2} \tag{10}$$

Where $m_{J,0}$ is the Jeans mass at mean number density $n_0$.

### 3.4 Alfven Waves

Alfven waves are magnetohydrodynamic waves in the same way that sound waves are thermal waves. We can deduce that their velocity takes the form,

$$v_A = \sqrt{\frac{B^2}{\mu' \rho}} \tag{11}$$

where $\mu'$ is the permittivity.

## 4 Derivation

### 4.1 Shocked Gas

Assuming the magnetic pressure is greater than thermal pressure a gas cloud will be affected by shocks relative to the alfven velocity $v_A$. Consider a sheet of thickness $L$, density $\rho_1$ and magnetic field $B_1$ before being shocked and thickness $\lambda$, density $\rho_2$ and magnetic field $B_2$ after being shocked. The Alfven Mach Number is given by,

$$M_A = \frac{v}{v_A} = \frac{\rho_2}{\rho_1} = \frac{L}{\lambda} = \frac{B_2}{B_1} \tag{12}$$

Now consider a spherical cloud of post-shock gas that may become a prestellar core. Lets assume it has a radius the same as the thickness of the sheet. Hence its mass will be,

$$m \approx \rho_1 \lambda^3 = \frac{\rho_0 L^3}{M_A^2} = \frac{\rho_0 L_0^3}{M_{A,0}^2} \left(\frac{L}{L_0}\right)^{4-\beta} \tag{13}$$

where $M_{A,0}$ is the Alfven Mach number on the largest scale $L_0$.
We now assume that the number density of cores formed scales as $n \propto L^{-3}$. Hence,

$$n(m) dlog_{10}m \propto L^{-3} dlog_{10}m \propto m^{-3/(4-\beta)} dlog_{10}m \tag{14}$$

Inserting $\alpha = 0.4$ ($\beta = 1.8$) gives a value of $x = 1.36$; almost exactly a Salpeter slope.

### 4.2 Collapsing Cores

We must now consider the probability these cores will collapse to form stars. In order to do this we must first examine the probability density function of the number density. This has been found to be gaussian by numerical simulations (Padoan, Nordlund & Jones [9]),

$$p(lnn')dlnn' = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{1}{2}\left(\frac{lnn' - < lnn' >}{\sigma}\right)^2\right] dlnn' \tag{15}$$

where $n'$ is the number density divided by the mean number density. $p(n')dn'$ can be thought of as the density disribution per unit volume. If we want to consider the density distribution per unit mass we must multiply by a factor of $n'$. Hence the probability distribution of Jeans masses is,

$$q(m'_J)dm'_J = n' p(lnn') \frac{dlnn'}{dn'} \frac{dn'}{dm'_J} dm'_J \tag{16}$$

Where $m'_J$ is the Jeans mass divided by the Jeans mass at mean density. As,

$$m'_J = n'^{-1/2} \tag{17}$$

we can show that,

$$q(m'_J)dm'_J = \frac{2}{\sqrt{2\pi\sigma^2}} m_J'^{-3} exp\left[-\frac{1}{2}\left(\frac{2lnm'_J + < lnn' >}{\sigma}\right)^2\right] dm'_J \tag{18}$$

Hence the Jeans mass varies through the cloud so the concept of a local Jeans mass must be considered rather than the global Jeans mass of the cloud. In order for a core to collapse it must have a mass greater than the local Jeans mass. The probability of this is the integral of the probability distribution for the Jeans mass,

$$\int_0^{m/m_{J,0}} q(m'_J) dm'_J \tag{19}$$

We can insert this factor into our mass function to yield,

$$n(m) dlog_{10}m \propto m^{-3/(4-\beta)} \left[ \int_0^{m/m_{J,0}} q(m'_J) dm'_J \right] dlog_{10}m \tag{20}$$

This introduction of the probability of collapse gives us the flattening off in the Mass Function we require to match observations.

## 5 Problems with this Theory

There are two main problems with this theory, The first is that in finding that $N \propto L^{-3}$ we assume that the two different scales have the same Mach number, this is incorrect as the Mach number is scale dependent. Also there is no treatment of magnetic pressure in the collapsing cores. While one is presented in Padoan & Nordlund [8] it is still very sketchy.

## References


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