Wavelets: Applications in Astrophysics

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1 Introduction

The aim of this pedagogical seminar is to encourage intuitive understanding of wavelet analysis. I propose to do this by first covering the mathematical foundations of the technique and then will illustrate the different applications using real-life astrophysical examples. I hope this will not only make it possible to use wavelets in your own research but also mean that you can critically assess other work which uses wavelet results.

2 Definitions

The Morlet-Grossman definition (Grossman and Morlet, 1984) for the continuous wavelet transform (CWT) of a 1D signal $f(x)$ is

$$W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\psi^*\left(\frac{x-b}{a}\right) dx$$

where $W(a,b)$ is the wavelet coefficient of the function $f(x)$ using the wavelet $\psi(x)$ of scale a at position b. It should be noted that in most cases a 2D image can be assumed to be separable and so the 1D transform can be applied on one row at a time, which becomes $f(x)$.

The signal $f(x)$ can be restored from $W(a,b)$ using the inverse transform

$$f(x) = \frac{1}{C_\chi} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{a}} W(a,b)\chi\left(\frac{x-b}{a}\right) \frac{dadb}{a^2}$$

where

$$C_\chi = \int_0^{+\infty} \frac{\tilde{\psi}^*(\nu)\tilde{\chi}(\nu)}{\nu} d\nu = \int_{-\infty}^{0} \frac{\tilde{\psi}^*(\nu)\tilde{\chi}(\nu)}{\nu} d\nu$$

Here tildes mean the Fourier transform. Usually $\chi(x) = \psi(x)$ but can be chosen to be other functions to enhance certain features.

Admissibility Condition C_χ must be defined if reconstruction is to be possible. If $\chi = \psi$ then this leads to $\tilde{\psi}(0) = 0$, i.e. the mean of the wavelet function ψ is zero.

Vanishing Moments It is also desirable that a wavelet has as many vanishing moments as possible to ensure that it is well defined in time and frequency space. If its Fourier transform is p times differentiable then the wavelet has p vanishing moments.

3 Some Wavelet Transform Properties

3.1 Linear

$$\text{If } f(x) = f_1(x) + f_2(x) \text{ then } W_f(a,b) = W_{f_1}(a,b) + W_{f_2}(a,b)$$

3.2 Covariant under translation

$$\text{If } f_0(x) = f(x - x_0) \text{ then } W_{f_0}(a,b) = W_f(a, b - x_0)$$

3.3 Covariant under dilation

$$\text{If } f_s(x) = f(sx) \text{ where } s \text{ is a scalar then } W_{f_s}(a,b) = \frac{1}{\sqrt{s}} W_f(sa, sb)$$

4 Common Wavelet Functions

Mexican Hat The second derivative of a Gaussian, $\psi(x) = (1 - x^2)e^{\frac{-x^2}{2}}$

Haar

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Morlet This is a complex wavelet therefore has real and imaginary parts

$$\psi_r(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \cos(2\pi\nu_0 x) \ , \ \psi_i(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \sin(2\pi\nu_0 x)$$

where ν_0 is a constant. It should be noted that this wavelet is not admissible. However, for ν_0 greater than around 0.8 the mean is close enough to zero such that reconstruction is acceptable.

5 Discrete Wavelet Transform

For use on real digital images this transform has to be modified by discretizing the wavelet. The increase in scale can be thought of as the function $f(x)$ being projected on to a grid of increasing spacing, which goes as $a = 2^j$, with j the *scale number*. This is done through a scaling function $\phi(x)$ which is dilated and translated before the scalar product with the function $f(x)$ is found. The scalar products created by this projection are known as $c_j(k)$:

$$c_j(k) = \langle f(x), 2^{-j}\phi(2^{-j}[x-k])\rangle$$

A necessary quality of scaling functions ϕ is

$$\frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum_l h(l)\phi(x-l)$$

This is very useful since, once we have knowledge of the $h(l)$, we can obtain all the coefficients $\{c_j\}$ from the set c_0, without having to directly compute any more scalar products. What is actually desired, however, is the set of wavelet coefficients for all scales $\{w_j(k)\}$. I have described the recipe for the *à trous* algorithm (Holschneider et al., 1989; Shensa, 1992).

Recipe for finding wavelet coefficients

i) Specify the scaling function $\phi(x)$
ii) Get $h(l)$ from the scaling function dilation relation
iii) Get $\{c_j(k)\}$ from $h(l)$ using $\{c_0(k)\}$ and the recursion relation for $c_j(k)$
iv) Get wavelet coefficients $\{w_j(k)\}$ from the difference between successive scales of c_j

6 Applications

There are many areas of study in astrophysics for which can wavelets can be useful. The applications can be grouped roughly as follows

- Periodicity Analysis
- De-noising
- Prediction and Monitoring
- Characterisation
- Compression
- Detection

There are many books to choose from if you want to find out about wavelets in great mathematical detail. An excellent place to start is, for example, Starck et al. (1998)). There are also many resources online (e.g. www.wavelet.org or www.amara.com).

References

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