The magnetohydrodynamics (MHD) equations and the physics of accretion disks and jets

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1 Introduction

Plasmas appear very often in Astronomy, and most of us have the idea that 'they have to do with ionised matter', which is actually true. However, we need to precise this concept a little more if we want to study how plasmas behave in general, and how astrophysical plasmas do it. The aim of the first part of this lecture is to give a brief introduction to plasma physics, starting from very first principles, and show how the magnetohydrodynamics (hereafter MHD) equations can be constructed. As we shall see afterwards, the MHD equations allow for the macroscopic description of plasmas and, although they are only valid under some restrictive conditions, they can be used to study several physical phenomena, such as equilibrium, propagation of waves and instabilities. In the second part of the lecture, we are going to see how the MHD theory is used to explain some important astrophysical problems, as the the equilibrium in accretion disks and the formation of jets.

2 Basic concepts

Plasmas are composed by partially or totally ionised matter. The theory of partially ionised plasmas is quite more complicated than the theory for totally ionised ones, so we are going to restrict our description to the latter. We are going to deal with two different particle species: the ions and the electrons. Any charged particle within the plasma attracts other particles with opposite-sign charge and repels particles with the same-sign charge. In a gas, the collisions are simply due to the random motion of the particles. In a plasma, instead, the collisions are governed by electrostatic forces. If each charge were isolated in vacuum, the electrostatic potential generated by each particle would be simply $\Phi_C = Ze/4\pi\epsilon_0 r$ (Coulomb potential), where ϵ_0 is the permittivity of the vacuum. However, the presence of other charged particles modifies this potential producing a screening effect. Let's calculate the effective electrostatic potential produced by a charged particle inmersed in a plasma. If each species in the plasma is in thermal equilibrium, the distribution functions for both ions and electrons are given by a Maxwellian functions of the form:

$$f(\mathbf{r}, \mathbf{v}) \propto \exp\left(-\frac{mv^2}{2k_BT} + \frac{e_j\Phi}{k_BT}\right), \tag{1}$$

where k_B is the Boltzmann constant. Φ is our unknown, the effective potential felt by a particle of the species j (j = ion, electron) with charge e_j and generated by all the other charges surrounding it. This potential must satisfy Poisson's equation:

$$\nabla^2\Phi = \frac{1}{\epsilon_0}\rho(r), \tag{2}$$

where $\rho(r)$ is the electric charge density $\rho(r) = \sum_j e_j n_j(r)$. The particle density for each species $n_j(r)$ can be obtained integrating eq. (1) in the velocity space, which gives $n_j(r) = n_0 \exp(e_j\Phi(r)/k_BT)$. Inserting the expression of $\rho = \rho(\Phi, r)$ into the Poisson's equation (2), we obtain a differential equation for the electrostatic potential Φ. If we assume that $e\Phi/k_BT << 1$, we see that the solution of the Poisson's equation is given by

$$\Phi \propto \frac{1}{r}\exp(-r/\lambda_D). \tag{3}$$

This means that the effective potential produced by a charged particle screened by other electric charges decays at a characteristic distance λ_D, which is known as the *Debye length.* To obtain this solution we have made the important assumption that our medium is electrically neutral, i.e., the total positive charge balances the total negative charge. In the case of a plasma composed by electrons and protons, the Debye length is $\lambda_D = (\epsilon_0 k_B T/2n_0e^2)^{1/2}$. The existence of such a cutoff in the electrostatic potential means that each particle will be only able to interact with particles present within a radius given by the Debye length. Applying all this, we can try to improve our definition of plasmas. *A plasma is a medium composed by two species charged with opposite sign, but electrically neutral as a whole, in which the electrostatic forces are only important to determine the* **collective** *behaviour of the particles.* We can see that, for this to be valid, we need that the dimensions of our system have at least the Debye length.

3 The construction of the MHD equations

In this section we are going to outline the steps of the construction of the MHD equations. A particle with charge q inmersed in an electromagnetic field has a motion governed by the Lorentz force:

$$\mathbf{F} = \mathrm{q}(\mathbf{E} + \mathbf{v}\mathrm{x}\mathbf{B}). \tag{4}$$

However, the quantity of particles in a plasma is very large to try to solve the equations of motion for each of them individually. As usual in these cases, the best approach is to use statistical mechanics. Let's assume we have N particles in the plasma, each one characterised by its position $\mathbf{r}_i$ and its velocity $\mathbf{v}_i$ $(i = 1, \ldots, N)$, which are functions of the time t. We can make the description of our system in a space of six dimensions (the three spatial coordinates and the three velocity coordinates). The number of particles with coordinates ($\mathbf{r}$,$\mathbf{v}$) in the time t in a volume element $dx\, dy\, dz\, dv_x\, dv_y\, dv_z$ is given by

$$f(\mathbf{r}, \mathbf{v}, t)\, dx\, dy\, dz\, dv_x\, dv_y\, dv_z, \tag{5}$$

where $f(\mathbf{x}, \mathbf{v}, t)$ is the so-called *distribution function* of the system. After a time which is long compared with the mean time between collisions of particles of a same species, the plasma can be considered as composed by two fluids which are individually thermalised. In that case, the distribution function for each species in our plasma becomes a Maxwellian. Considering that we are dealing with an isotropic system, we have :

$$f(\mathbf{r}, \mathbf{v}, t) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-v^2/v_T^2\right), \tag{6}$$

where $v_T = (2k_B T/m)^{1/2}$. It is important to note that until now we have not included at all the fact that we are working with charged particles. This description is completely generic for any gas of thermalised particles.

If we do not have any source or sink of matter in our system and we ignore collisions (just for a while), the rate of change of the number of particles in any volume V must equal the net flux of particles into that volume. This means that:

$$\frac{\partial}{\partial t}\int_V f \mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{v} = -\int_V f\,\mathbf{v}\cdot\mathrm{d}\mathbf{S}\,\mathrm{d}^3\mathbf{v} - \int_V f\,\mathbf{a}\cdot\mathrm{d}\mathbf{S}_\mathbf{v}\mathrm{d}^3\mathbf{r}. \tag{7}$$

The right hand of this equation has two terms, one gives the contribution of the flux in the spatial coordinates and the other one gives the contribution in the velocity coordinates. If we use Gauss theorem and we do our volume arbitrarily small, we arrive to the following equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_\mathbf{r} f + \frac{\mathbf{F}}{m}\cdot\nabla_\mathbf{v} f = 0, \tag{8}$$

where we have used the fact that $\mathbf{a} = \mathbf{F}/m$, which we consider independent of the velocity $\mathbf{v}$. Equation (8) is known as the *collisionless Boltzmann equation*, and it says that, in the absence of collisions, the convective derivative of f is 0. Actually, this is just a consequence of a more fundamental mathematical theorem known as Liouville's theorem.

To solve an equation of the kind of (8), we need to know which forces act on our system. And now for the first time in this formalism we introduce the fact that we are working with a plasma. As we have mentioned before, the force acting on each particle of a plasma is the Lorentz force, whose expression is given by eq. (4). If we replace (4) into the Boltzmann equation (8), we obtain the so-called *Vlasov equation*. But the Vlasov equation, as well as Boltzmann equation, does not include the fact that there are collisions in the plasma. To include this fact, we need to add explicitly a collision term into the Vlasov equation, which becomes:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_\mathbf{r} f + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_\mathbf{v} f = \left(\frac{\partial f}{\partial t}\right)_c \tag{9}$$

Equation (9) simply says that the convective derivative of the distributon function f is actually given by the rate of change of f due to collisions. The explicit calculation of the collision term is beyond the scope this lecture, so we are only going to mention its consequences when necessary.

The usual way of recovering macroscopic magnitudes starting from the microscopic equations in statistical mechanics is taking *moments* of the Boltzmann equation. This means to integrate eq. (8) in the velocity space, premultiplied by different powers of the velocity vector $\mathbf{v}$. In our case, we should do this with eq. (9), and this will allow us to obtain the macroscopic equations which govern each species present in a plasma (the *two-fluids equations*). We are not going to do these calculations explicitly. We are going to show diretly under which assumptions we can reunite the two-fluids equations in order to describe our system as a single fluid. Let's define the following macroscopic parameters for a plasma:

- mass density:
$$\rho_m = n_i m_i + n_e m_e \approx n m_i \tag{10}$$
which uses the fact that the mass of the ions m_i is much larger than the mass of the electrons m_e. The number of ions $n = n_i$ is equal to the number of electrons n_e in the case of a plasma composed by protons and electrons.

- velocity of the fluid:
$$\mathbf{v} = \frac{1}{\rho}(n_i m_i \mathbf{v}_i + n_e m_e \mathbf{v}_e) \approx \mathbf{v}_i \tag{11}$$
which means that we consider that the momentum in the plasma is mainly carried by the ions.

- current density:
$$\mathbf{J} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e) = ne(\mathbf{v}_i - \mathbf{v}_e), \tag{12}$$
when $n_i = n_e$.

Rearranging the two-fluids equation using the definitions given above, we obtain the *single-fluid equations*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{(continuity)} \tag{13}$$
$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \mathbf{J}\times\mathbf{B} - \nabla p + \mathbf{F} \qquad \text{(force balance)} \tag{14}$$
$$\frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t} = \mathbf{E} + (\mathbf{v}\times\mathbf{B}) - \frac{1}{ne}(\mathbf{J}\times\mathbf{B}) + \frac{1}{ne}\nabla p_e - \eta\mathbf{J}. \qquad \text{(Ohm's law)} \tag{15}$$

In eqs. (14) and (15), p_e refers to the presure produced by the electrons and p to the total pressure produced by the two species $p = p_e + p_i$. These terms involving pressures are the consequence of taking into account the collisions in eq. (9). The $(\mathbf{J}\times\mathbf{B})$ term in eq. (15) describes the Hall effect and the term with ∇p_e describes the fact that electron pressure gradients will drive currents in the plasma. $\mathbf{F}$ in eq. (14) represents any possible external force. We emphasize again that all this deductions are valid if the plasma is collision dominated, i.e., the mean free path for collisions must be much smaller than the Debye length of our system. This guarantees that both the fluid of electrons and that of ions are thermalised, and both fluids can be described as only one.

Let's see now under which assumptions the single-fluid equations (13) to (15) can be simplified for giving the MHD equations. If the currents J in the plasma are small in such a way that $(v_i - v_e) << v_i$, the term of the Hall effect in eq. (15) can be neglected. Besides, if we restrict to only those phenomena with low frequencies compared to both the *plasma frequency* $\omega_p = (ne^2/\epsilon_0 m_e)^{1/2}$ and the ion *cyclotron frequencies* $\omega_{ci} = eB/m_i$, it can be demonstrated that both the left-hand side of eq. (15) and the term with ∇p_e are negligible compared to the first two terms of the right-hand side. Under these assumptions we finally can simplify the single-fluid equations to obtain:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{16}$$

$$\rho \frac{d\mathrm{v}}{dt} = (\mathbf{J} \times \mathbf{B}) - \nabla p \tag{17}$$

$$\mathbf{E} + (\mathbf{v} \times \mathbf{B}) = \eta \mathbf{J}, \tag{18}$$

to which we add the Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (\mathrm{Faraday's\,law}) \tag{19}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad (\mathrm{Amp\grave{e}re's\,law}) \tag{20}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{21}$$

and an equation of state

$$\frac{p}{\rho^\gamma} = \mathrm{const.}. \tag{22}$$

The system of coupled equations (16) to (21) are known as the *resistive MHD equations*. If we set $\eta = 0$, we have the *ideal MHD equations*.

4 Equilibrium - axisymmetric systems

One of the most important applications of the MHD equations is the study of equilibria in plasmas. In particular, we will be interested in the equilibria of *stationary* plasmas. Nonstationary equilibria do exist, but tend to be less stable to small perturbations than their static counterparts. In the stationary state, we need to solve a simplified version of the MHD equations:

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{23}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{24}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{25}$$

From equation (23) we can see that $\mathbf{B} \cdot \nabla p = 0$, which means that, for a stationary plasma, the pressure is constant along the magnetic field lines. Put another way, the magnetic field lines lie on isobaric surfaces.

Let's see what happens if we solve the system of eqs. (23) to (25) for a plasma configuration with axial symmetry. The eq. (25) can be written in cylindrical coordinates (r, ϕ, z) in the following way:

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial (rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} = 0. \tag{26}$$

$\mathbf{B}_\phi$ is usually called the *toroidal magnetic field* and $\mathbf{B}_p = B_r \hat{e}_r + B_z \hat{e}_z$ is the *poloidal magnetic field.* The axial symmetry $(\partial/\partial\phi \equiv 0)$ allows us to introduce a flux function ψ to represent the field components of the poloidal plane, yielding

$$\mathbf{B} \equiv \mathbf{B}(r,z) = \frac{1}{r}\nabla\psi \times \hat{e}_\phi + B_\phi \hat{e}_\phi. \tag{27}$$

If we replace the form for the magnetic field given by eq. (27) into the Ampère's law (eq. (24)), we can obtain an expression for the electric current $\mathbf{J}$ as a function of the flux ψ and the toroidal magnetic field B_ϕ. If we introduce this into the eq. (23), we can finally obtain the equilibrium equation for an axisymmetric configuration of plasma:

$$\nabla^2\psi = -\mu_0 r^2 \frac{dp(\psi)}{d\psi} - \mu_0^2 g(\psi)\frac{dg(\psi)}{d\psi}, \tag{28}$$

where $g = rB_\phi/\mu_0$. Eq. (28) is known as the *Grad-Shafranov equation.*

5 Applications to Astrophysics: the dynamics of accretion disks and jets

The MHD theory can be applied to the study of a wide variety of problems in Astrophysics. In this lecture, we are going to concentrate our attention in the dynamics of accretion disks and jets. Accretion disks and high speed outflows are observed in different astronomical sources, from protostars in the solar neighbourhood, to galactic X-ray binaries and active galactic nuclei (AGN). Although some of the models we are going to discuss here can be applied to any of these objects, we are going to focus in particular in the accretion disks and jets of AGN. As we shall see in the next sections, the equilibrium of accretion disks and the formation of jets are two problems intimately linked.

5.1 The dynamics of accretion disks

Let's consider a typical AGN, with a disk of matter accreting onto a central black hole. The trajectory of a particle far enough from the black hole is given by the laws of classical mechanics. However, when the particle gets nearer, relativistic effects must be taken into account to determine its motion in the gravitational field of the central object. General relativity teaches us that the last stable orbit for a particle approaching a spherically symmetric black hole has a radius $R = 6GM/c^2$, where M is the mass of the black hole. This means that any particle which loses energy after this orbit will inevitably fall inwards. However, we are not going to worry about what happens to the particle beyond the marginally stable orbit, but how this last stable orbit is reached. A particle in the accretion disk of an AGN needs to lose both energy and angular momentum to approach to the black hole. The energy is basically lost through the emission of radiation. But what happens with the angular momentum? For a black hole of mass $10^8 M_\odot$, the specific angular momentum in the marginally stable orbit is $\sim 10^{24} cm^2 s^{-1}$. For matter orbiting a typical galaxy, the specific angular momentum is $\sim 6.10^{28} cm^2 s^{-1}$. In binary systems, the angular momentum lost by the accreting material can be transmitted outwards through the same accretion disk to the donor source. However, this mechanism does not seem very plausible for AGN, for which we do not know the source of the mass which enters the accretion disk. Different mechanisms have been proposed for the dissipation of angular momentum. Most of them assume that the disks around AGN are thin ($z << r$):

1) **torques due to viscosity**: although this seems to be a reasonable explanation, it is very difficult to imagine a mechanism which produces the high viscosities needed to account for the total loss of angular momentum in the accretion disks of AGN.

2) **torques due to magnetic stresses - the Balbus-Hawley instability**: this instability has been originally described by Velikhov (1959) and Chandrasekhar (1981), and it was rediscovered by Balbus and Hawley in 1991, who proposed this as a way of dissipating anguar momentum in an accretion disk. If the accretion disk is threaded by an initial poloidal magnetic field and this field suffers radial perturbations, an instability is generated, which grows as fast as the orbital frequency. The study of instabilities in a plasma constitutes a whole branch of the MHD theory. Basically, the idea is to solve the MHD equations after applying small perturbations to the magnetic field $\mathbf{B}$ and the velocity $\mathbf{v}$. We obtain a dispersion relation for the possible waves produced as a consequence of the perturbation. In some cases, these waves can grow exponentially, producing instabilities. The Balbus-Hawley instability, in particular, appears in plasmas when the *Alfvén velocity* v_A is smaller than the speed of sound c_s in the medium [1]. It has been demonstrated that this mechanism can be at least as efficient as viscosity to remove angular momentum from the accretion disks.

3) **Magnetised winds**: Blandford and Payne (1982) showed that an accretion disk initially threaded by a poloidal magnetic field can lose energy and angular momentum by the ejection of matter which is spun up to velocities greater than escape. Let's suppose that initially we have an accretion disk axisymmetric, with infinite conductivity, which is a cold MHD fluid. By 'cold' we mean that that the magnetic forces are much more important than the thermal effects in the plasma. In this case, we say that the magnetic field lines are 'frozen' in the plasma, and the particles of the plasma are forced to move along the magnetic field lines ($\mathbf{J} \times \mathbf{B} = 0$, i.e. $\mathbf{J} // \mathbf{B}$). Put another way, the condition of cold plasma means that the Alfvén velocity v_A is much larger than the sound speed c_s. On the other hand, if the velocity of sound c_s is also much smaller than the orbital speed v_s , the field lines are forced to rotate with the orbital speed of the point where it passes through the disk midplane. Each particle of gas in the disk feels an effective potential given by

$$\Phi_{\text{eff.}} = -\Omega^2(r_0) \left[\frac{1}{2} r^2 + \frac{r_0}{\sqrt{r^2 + (r - r_0)^2 cot^2 \theta}} \right], \tag{29}$$

where r_0 is the radial coordinate of the intersection point between the outcoming poloidal field line and the disk midplane, and θ is the angle between the field

[1] The Alfvén velocity is a characteristic magnitude of plasmas and is defined as $v_A = B/\sqrt{4\pi\rho}$. This velocity measures how important the magnetic field is in comparison to the gas pressure in the plasma.

line and the rotational axis. A particle is in equilibrium when $r = r_0$, where the first derivative of the effective potential $\Phi_{\rm eff.}$ vanishes. However, when the second derivative of $\Phi_{\rm eff.}$ is negative, the equilibrium is unstable. This is the situation for $\theta > 30°$, because the projection of the gravitational force onto the field line is insufficient to balance the centrifugal force felt in the rotating frame. In this case, the particles are accelerated outwards from the accretion disks. This picture holds until the velocity of the particles reaches the Alfvén velocity v_A.

The arguments given above constitute a simplified explanation of the model and results obtained by Blandford and Payne (1982). What they actually do is to solve the Grad-Shafranov equation (i.e.a stationary plasma is supposed), imposing as initial condition that the trajectories of the particles in the accretion disk are determined by the gravitational potential of the central black hole. The solutions of the Grad- Shafranov equation under these conditions are different in three different regions of the velocity space of the particle. However, we are going to leave this discussion for the next section, where we treat the problem of the formation and sustainment of jets.

5.2 The formation of jets

From the discussion of magnetised winds of the preceding section, we can see that the formation of jets is a process strongly linked to the dynamics of accretion disks. The study of the formation of jets aims to answer two basic questions: what mechanism *accelerates* the jet and how the jet gets *collimated.* Basically, there are three families of models which try to explain this process:

5.2.1 Hydromagnetically driven jets

The first family of models propose that the jet is formed for the extraction of matter carrying energy and angular momentum from the accretion disks. The model by Blandford and Payne (1982) explained above is a typical example of how hydromagnetical jets can be formed. As we have already seen, such a model proposes that kinetic energy is extracted from the initial poloidal magnetic field to accelerate the particles in the plasma in the axial direction. The accelerating force in this case is the centrifugal force, but this mechanism is only possible if there is an initial magnetic field. The first regime of solutions of the Grad-Shafranov equations corresponds to velocities of the particles below the Alfvén velocity v_A. During this stage the jet is formed. The collimation is due to the induced toroidal magnetic field. Although initially the magnetic field is considered to have only a poloidal component, the motion of charged particles along the poloidal field lines creates a poloidal electrical current. This poloidal current induces a toroidal component of the magnetic field. This phenomenon is known as *dynamo effect.* In the second regime, after that the particles get velocities larger than v_A, the induced toroidal magnetic field becomes more important than the poloidal magnetic field. At this stage the outflow reaches its maximum degree of collimation. However, it is not clear for how much time this situation can be stable. Different instabilities can develop in plasma jets which eventually make fail these idealised models.

5.2.2 Electromagnetically driven jets

This second family of models propose as well that the energy necessary to form the jet comes from the accretion disk. However, the energy and angular momentum extracted from the disk are carried by the same electromagnetic field, and not by matter. These kinds of models have been pioneered independently by Blandford (1976) and Lovelace (1976) and more recent works have been made by Lovelace et al. (2002), among others. The so-formed jets are in general referred as *Poynting jets*. These models have some of the same initial conditions that the hydrodynamics models have: an accretion disk with a cold stationary plasma is considered. Again, the disk is initially assumed to be threaded by a poloidal magnetic field. However, these models in general assume also the presence of a force-free magnetosphere surrounding the disk, which is composed by particles that might have been removed from the accretion disk. Then, the Grad-Shafranov equation is solved for the whole system (disk + magnetosphere), with different possible boundary conditions. As result, the magnetosphere appears to be exerting an electromagnetic torque on the disk, which is able to remove both energy and angular momentum through the *Poynting vector* $\mathbf{S} \propto (\mathbf{E} \times \mathbf{B})$.

These models of Poynting outflows have some advantages to explain the formation of jets in AGN in comparison to the hydrodynamical models. First of all, the hydrodynamical models only predict the expulsion of matter when the angle of the poloidal field with the symmetry axis is larger than $30°$. This condition does not appear in the pure electromagnetical models, so the extraction of angular momentum is still possible at smaller angles. A second advantage, even more important than the first one, is that Poynting jets move at the speed of light c. Observations with Very Long Baseline Interferometry (VLBI) show that the AGN jets have relativistic velocities, with Lorentz factors $\Gamma \sim 10$ or more. So it is quite likely that the formation of AGN jets is mainly due the extraction of angular momentum by pure electromagnetic forces.

5.2.3 Jets with energy and angular momentum extracted from rotating black holes

In this section we summarise the concepts of how a rotating black hole behaves when it is threaded by a magnetic field. We follow the digest made in the book by Krolik, leaving a rigurous treatment for general relativity courses.

By the same definition of event horizon of a black hole, any electromagnetic wave approaching it must be completely absorbed with zero reflection. Thus, it makes sense to consider that the event horizon acts like free space for the propagation of electromagnetic waves. It can be demonstrated that a black hole has the same resistivity as an open wave guide: $R = 377\Omega$. Let's consider, in particular, a black hole rotating around the z axis. We can consider that the distortion of the space-time produced in the vicinity of such a body is equivalent to that of a spherically symmetric black hole, but with a 'dragging' of the distorion due to rotation. If the black hole is inmersed in a magnetic field (which we can consider as constant), the frame-dragging induces and electromagnetic force (EMF) in the same way it happens with conducting wires forced to move through a magnetic field. Actually, the induction of an EMF in this case is due to the interaction between the gravitational and the electromagnetic fields. So the rotating black hole inmersed in a magnetic field behaves as a battery. The magnitude of the induced EMF can be calculated using Ampère's law, yielding EMF $\propto 10^7(M/M_\odot)B\,V/G$, where M is the mass of the black hole and B, the magnitude of the magnetic field in Gauss. Thus, it is possible to extract both energy and angular momentum from the rotating black hole.

This argument was formally demonstrated by Blandford and Znajek in 1977. If a black hole is inmersed in a strong magnetic poloidal magnetic field whose lines close at infinite distance, and rotates enough rapidly, the medium surrounding the hole is unstable. This happens because any charged particle in the medium will be electrostatically accelerated and will radiate, producing even more particles in the form of electron-positron pairs. In the vicinity of the black hole horizon, the electromagnetic field behaves as force-free. Thus, Blandford and Znajek solve the relativistic equivalent to MHD equations in this force-free magnetosphere surrounding the black hole. They show that the maximum induced toroidal field B_t is proportional to both the poloidal field B_p and the black hole angular velocity Ω. The maximal rate at which energy can be generated corresponds to the case in which the induced toroidal field B_t is maximum:

$$L_{BZ}(max) \sim \left(\frac{B_{ph}B_{th}}{4\pi}\right)\pi R_h^2\left(\frac{R_h\Omega_h}{c}\right)c. \tag{30}$$

This is known as the *Blandford-Znajek luminosity.* The subindex h indicates that the different magnitudes are evaluated in the horizon event.

The Blandford-Znajek mechanism is the most widely accepted model for the formation of jets. Let's emphasize that both in this model and as in the models of section 5.2.2, the initial jet is not composed by matter at all. It is the Poynting vector which is removing energy and angular momentum. However, this Poynting vector can eventually form a plasma jet, either by creating pairs of electrons and positrons, either sweeping and collimating electrons and protons encountered in its way out the central source of the AGN.

The Blandford-Znajek model predicts the extraction of energy and angular momentum independently of the existence of an accretion disk. If an accretion disk is present, the mechanism of extraction of energy from the black hole competes with the extraction of energy from the disk, and it has been recently argued that the latter can be even more efficient for the most accepted models of the accretion disks (see Livio et al., 1999).

6 Some final comments

The aim of this lecture has been to show how the MHD equations can be constructed from first principles, and how they can be applied to study specific astrophysical problems as the dynamics of acretion disks and jets. Some of the main models have been reviewed, particularly those that treat the accretion disks and jets in the stationary state. There is another approach, consisiting in the numerical solution of the time-dependent MHD equations, which we have not mentioned here (e.g., Shibata and Uchida, 1985 and 1986, or Bell and Lucek, 1995) . In general, these models fail in reproducing the behaviour of jets in large scales, although they are able to reproduce the evolution of the jet during the formation stage. Both the steady-state and the time-dependent approaches are still far from being definitive. MHD plasmas are known to suffer from different kinds of instabilities, which we barely mentioned during this lecture. These instabilities are often referred as arguments against the validity of the most accepted models.

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[2] I haven't found this book particularly good, but it is the only book in the ROE library which more or less covers the fundamentals of plasma physics. Although I haven't used it to prepare this lecture, I should better recommend the book by Krall and Trivelpiece, *Principles of Plasma Physics*, McGraw-Hill, 1973