Magnification bias and applications to the cluster lens CL0024+1654

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Abstract

A rather new aspect of weak gravitational lensing has arisen from the fact that the apparent distribution of background sources on the sky is affected by the gravitational lensing properties of the intervening matter (e.g. clusters and large-scale structures), provided the slope of the background galaxy number counts differs from a critical value corresponding to a balance between the dilution due to local stretching of the background sky and the density enhancement due to magnification of faint sources above the observed magnitude limit. This effect, named the magnification bias effect, was first pointed out by Broadhurst et al. (1995) and suggested as a new method for measuring the projected mass distribution of galaxy clusters.

1 Fundamentals of gravitational lensing

In this section we shall summarize the gravitational lensing theory in the limit where geometrical optics can be applied. This approximation is sufficient for the treatment of gravitational lensing by galaxy clusters, where the gravitational fields are weak and stationary, and the deflecting mass is localized in a region small compared to the distance between source and deflector, and between deflector and observer. Therefore, the lens can safely be approximated by a single plane of matter with a projected surface mass density Σ. We refer to this as *the thin lens approximation.*

The lens equation. The deflection of a light ray by a gravitational lens is described by the so-called *deflection angle*, $\hat{\boldsymbol{\alpha}}$, which depends on the projected mass distribution of the lens, $\Sigma(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the impact vector of the light ray. The deflection is given by (see e.g. Schneider et al. (1992))

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2}\int \Sigma(\boldsymbol{\xi}')\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}d^2\boldsymbol{\xi}'. \tag{1}$$

Letting $\boldsymbol{\eta}$ denote the position vector of the source in the source plane and recalling the definition of angular-diameter distance one can write down a relation between the true position of the source, and its observed position in the lens plane

$$\boldsymbol{\eta} = \frac{D_s}{D_d}\boldsymbol{\xi} - D_{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{2}$$

where D_d, D_s, and D_{ds} are the angular diameter distances of the lens, the source and between the lens and source, respectively. If more than one solution $\boldsymbol{\xi}$ exists for a fixed source position $\boldsymbol{\eta}$, then the lens will produce multiple images of the source on the sky. This is called strong lensing. A sufficient but not necessary criteria for a lens to produce multiple images of the same source can be quantified by the dimension-less surface mass density (also called *the convergence*)

$$\kappa(\boldsymbol{\xi}) = \frac{\Sigma(\boldsymbol{\xi})}{\Sigma_{cr}}, \tag{3}$$

where $\Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}}$ is *the critical surface mass density.* If somewhere in the lens plane $\kappa \geq 1$ then multiple images of some source position will be produced. Hence, Σ_{cr} is a characteristic value of the surface mass density which distinguishes between weak and strong gravitational lensing. Introducing a so-called *deflection potential*, ψ, as follows

$$\psi(\boldsymbol{\xi}) = \frac{1}{\pi}\int \kappa(\boldsymbol{\xi}') \ln|\boldsymbol{\xi} - \boldsymbol{\xi}'| d^2\boldsymbol{\xi}', \tag{4}$$

the following simple relations are obtained: $\boldsymbol{\alpha} = \frac{D_{ds}}{D_s}\nabla\psi$ and $\nabla^2\psi(\boldsymbol{\xi}) = 2\kappa(\boldsymbol{\xi})$; the latter being Poisson's equation in the lens plane.

Magnification, shear and convergence. Introducing the dimensionless coordinates $\boldsymbol{y} = \boldsymbol{\eta}/\eta_o$ and $\boldsymbol{x} = \boldsymbol{\xi}/\xi_o$, where η_o is a characteristic scale in the source plane, and $\xi_o = \eta_o D_d/D_s$, we find that the lens equation can be written in the simple form $\boldsymbol{y} = \boldsymbol{x} - \nabla\psi(\boldsymbol{x})$. The Jacobian of the lens mapping $\boldsymbol{y} \mapsto \boldsymbol{x}$ is then easily seen to be $\mathcal{A}_{ij} = \delta_{ij} - \psi_{ij}$, where $\psi_{ij} = \partial^2\psi/\partial x_j \partial x_i$. Alternatively, we can write

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}, \tag{5}$$

where $\boldsymbol{\gamma}$ is the shear, and is given by $\boldsymbol{\gamma} = (\frac{1}{2}(\psi_{11} - \psi_{22}), \psi_{12})$.

It can be shown that gravitational lensing conserves surface brightness. Hence, if $I^S(\boldsymbol{y})$ is the surface brightness distribution in the source plane, the observed brightness distribution in the lens plane becomes $I(\boldsymbol{x}) = I^S(\boldsymbol{y}(\boldsymbol{x}))$. In the case where a source is much smaller than the angular scale on which the lens properties change, the lens mapping can locally be linearized, and we find for an image point $\boldsymbol{x_0}$, corresponding to a source point $\boldsymbol{y_0} = \boldsymbol{y}(\boldsymbol{x_0})$, the following linear relation

$$I(\boldsymbol{x}) = I^{(S)}\left[\boldsymbol{y_0} + \boldsymbol{\mathcal{A}}(\boldsymbol{x_0}) \cdot (\boldsymbol{x} - \boldsymbol{x_0})\right]. \tag{6}$$

According to this equation, the images of a circular source are ellipses. The ratios of the semi-axes of such an ellipse to the radius of the source are given by the inverse of the eigenvalues of $\boldsymbol{\mathcal{A}}(\boldsymbol{x_0})$, which are $1 - \kappa \pm |\boldsymbol{\gamma}|$, and the ratio of the solid angles subtended by an image and the unlensed source is the inverse of the determinant of $\boldsymbol{\mathcal{A}}$. The fluxes observed from the image and from the unlensed source are obtained by integrating over the brightness distributions $I(\boldsymbol{x})$ and $I^{(S)}(\boldsymbol{y})$, respectively, and their ratio is *the magnification* $\mu(\boldsymbol{x_0})$

$$\mu = \frac{1}{\det \boldsymbol{\mathcal{A}}} = \frac{1}{(1-\kappa)^2 - |\gamma|^2}. \tag{7}$$

The total magnification of an object is given by the surface brightness-weighted point-source magnification across the solid angle of the source. The images are thus distorted in shape and size. The shape distortion is due to the tidal gravitational field, described by the shear $\boldsymbol{\gamma}$, whereas the magnification is caused by both isotropic focusing by the local matter density κ and anisotropic focusing by shear (Schneider et al. (1992)). Note that since κ and $\boldsymbol{\gamma}$ depend on the source and lens redshifts via the angular diameter distances, so does μ.

2 The magnification bias effect

As shown above gravitational lensing not only distorts the shape of images but also magnifies the images, leaving the surface brightness invariant. For a fixed set of sources this will have an effect on the number density of images. Since the unit solid angle area of the unlensed source plane becomes a factor of μ larger due to lensing, the galaxy number counts within the unit solid angle of the lensed source plane will decrease by a factor of μ. Working against this is the increase of the galaxy number density, since fainter galaxies become brighter than the limiting magnitude due to the lensing magnification.

Let $n_0(>S, z)dz$ be the unlensed number density of galaxies with redshifts within dz of z and with fluxes larger than S. Then at an angular position, $\boldsymbol{\theta}$, where the magnification is $\mu(\boldsymbol{\theta}, z)$, the number counts are changed according to

$$n(>S, z) = \frac{1}{\mu(\boldsymbol{\theta}, z)} n_0 \left(> \frac{S}{\mu(\boldsymbol{\theta}, z)}, z \right). \tag{8}$$

(Schneider et al. (2000)). From eq. 8 it is seen that gravitational magnification can either increase or decrease the local number counts, depending on the shape of the unlensed number count function and on the behaviour of μ. The magnification bias effect is caused by this change in surface number density.

In general, the redshifts of the sources are not know, and so one can only observe the redshift-integrated counts

$$n(>S) = \int \mu^{-1}(z) n_0 \left(> \frac{S}{\mu(z)}, z \right) dz. \tag{9}$$

Observations show that the number counts of faint galaxies very closely follow a power law over a wide range of fluxes, i.e.

$$n_0(>S, z) = a \cdot S^{-\beta} p_0(z; > S), \tag{10}$$

where a is a constant and $\beta = -\frac{d \log n(>S)}{dS}$ is the slope of the power law, and depends on the wavelength band of the observations. $p_0(z; S)$ is the redshift probability distribution of galaxies with flux $> S$. From eq. 9 and eq. 10 one gets

$$n(> S) = a \cdot S^{-\beta} \int \mu^{\beta-1}(z) p_0(z; \frac{S}{\mu(z)}) dz, \tag{11}$$

and since $n_0(> S) = a \cdot S^{-\beta} \int p_0(z; S) dz = a \cdot S^{-\beta}$ we finally obtain the following expression for the ratio of the lensed and unlensed source counts

$$\frac{n(> S)}{n_0(> S)} = \int \mu^{\beta-1}(z) p_0 \left(z; \frac{S}{\mu(z)} \right) dz. \tag{12}$$

Note, that p_0 depends on z, and hence the lensed number counts do not strictly follow a power law in S. Since the redshift distribution $p_0(z, S)$ is currently unknown, the change of the number counts due to the magnification effect cannot be predicted. However, at very faint flux levels, the redshift distribution is likely to be dominated by galaxies at relatively high redshifts. Hence, for lenses at fairly small redshifts, $z_d \leq 0.3$ or so, we can approximate the redshifts-dependent magnification $\mu(z)$ by the magnification μ of a fiducial source at infinity. In doing so we get

$$\frac{n(> S)}{n_0(> S)} = \mu^{\beta-1}. \tag{13}$$

The first term of the exponent, β, describes the effect of flux magnification, while the last term, -1, describes the decrease in number density due to local stretching of the sky. Thus, a local estimate of the magnification can be obtained via eq. 13, given that one can measure the changes in the local number density of images and has a reliable value of the slope β. If the slope of the number counts is $\beta = 1.0$, the two effects will cancel exactly and there will be no magnification bias effect, while it will cause a decrease in the local number density for flatter slopes. In the last case the radial profile of galaxy number counts through a cluster is called the depletion curve. Note that since we from observations always will find $\frac{n(>S)}{n_0(>S)} > 0$, we can only measure the absolute value of μ, so we have to determine the image parity by some other means.

3 Estimating the cluster mass-distribution from magnification bias

As is obvious from eq. 7 one needs to know both the magnification, μ, and the shear signal, $|\boldsymbol{\gamma}|$, in order to derive the mass distribution κ. It would therefore be of great benefit if it was possible to find a relation between the shear and the convergence, since then the convergence could be derived directly from the depletion analysis.

In the weak lensing limit where $\kappa \ll 1$ and $|\boldsymbol{\gamma}| \ll 1$, we get the approximate relation $\mu \simeq 1 + 2\kappa$ which gives usa (using eq. 13)

$$\kappa = \frac{1}{2} \left(\left(\frac{n(> S)}{n_0(> S)} \right)^{1/(\beta-1)} - 1 \right). \tag{14}$$

However, the above estimate is only valid in outskirts of clusters where the surface density is low. If relations between κ and μ could be found which were valid beyond the weak regime, one could hope to obtain better estimates of the mass distribution.

Such relations between κ and μ have been studied by Dye and Taylor (1998) and van Kampen (1998). van Kampen (1998) finds that only two physically realistic local relations between μ and κ which result in single caustic, i.e. only one parity, solutions of eq. 8, and which furthermore are easily invertible. These one parity solutions are

$$\begin{aligned} \mu^{-1} &= (\kappa-1)^2 \qquad , (|\boldsymbol{\gamma}|=0) \qquad (15)\\ \mu^{-1} &= 1-2\kappa \qquad , (|\boldsymbol{\gamma}|=\kappa). \qquad (16) \end{aligned}$$

In the first case ($|\boldsymbol{\gamma}|=0$) there is no shear which means that the cluster is approximated by a single uniform sheet of matter, and the convergence estimator is easily seen to be

$$\kappa = 1-\mathcal{P}|\mu|^{-1/2} = 1-\mathcal{P}\left(\frac{n(>S)}{n_0(>S)}\right)^{\frac{-1}{2(\beta-1)}}, \qquad (17)$$

where $\mathcal{P}$ is the lens parity, i.e. the sign of μ. In the case of $\kappa=|\boldsymbol{\gamma}|$ one finds again the estimator of the convergence by inserting eq. 16 into eq. 7. This yields

$$\kappa = \frac{1}{2}-\frac{1}{2}\mathcal{P}|\mu^{-1}| = \frac{1}{2}-\frac{1}{2}\mathcal{P}\left(\frac{n(>S)}{n_0(>S)}\right)^{\frac{-1}{\beta-1}}, \qquad (18)$$

(van Kampen (1998)). Note that the critical radius occurs when $\kappa=\frac{1}{2}$, meaning that this estimator yields a smaller mass than the estimator in eq. 15. This is not a surprise to us since in the $\kappa=|\boldsymbol{\gamma}|$ case the shear contributes to the magnification while in the $|\boldsymbol{\gamma}|=0$ the convergence alone has to produce the magnification. Hence, one would expect κ to be larger in the latter case.

In an attempt to determine a more realistic mass-estimator van Kampen (1998) studied a sample of numerical clusters with known lensing properties from which he was able to derive an invertible *average* relationship between κ and $|\boldsymbol{\gamma}|$. As an adequate description van Kampen (1998) adopts

$$\mu^{-1} = (\kappa-c)(\kappa-c^{-1}) \qquad , \left(|\boldsymbol{\gamma}| = |1-c|\sqrt{\frac{\kappa}{c}}\right), \qquad (19)$$

where $0<c<1$. Clearly, eq. 19 has two critical lines, one at $\kappa=c$ and one at $\kappa=c^{-1}$. We can only measure $|\mu|$, and since $|\mu|$ can correspond to four different values of κ we can write down the estimator as

$$\kappa = \frac{c+c^{-1}}{2} - \mathcal{S}\left[\left(\frac{c+c^{-1}}{2}\right)^2 - \mathcal{P}|\mu|^{-1} - 1\right]^{1/2}, \qquad (20)$$

where $\mathcal{S}$ is a new parity giving the sign of $\kappa - \frac{c+c^{-1}}{2}$ (van Kampen (1998)). $\mathcal{S}$ indicates which side of the critical lines we are on. It is indeed a disadvantage of this estimator that two parities have to be set, since the sign of the magnification can not be inferred directly from observations. Note, that the $|\boldsymbol{\gamma}|=0$ approximation is recovered by setting $c=1$.

References

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