1 Introduction

Spherical geometry is one of the key fundamental frameworks of astronomy. The aim of this talk is to remind you of the basics, derive some key results, and demonstrate a couple of ways in which real astronomical problems can be solved.

I am not going to spend a great deal of time discussing the various astronomical coordinate systems. We shall begin in the abstract - discussing the key elements of geometry on the surface of a sphere. Various coordinate systems will then be presented as givens, within which we shall work.

2 Spherical Geometry

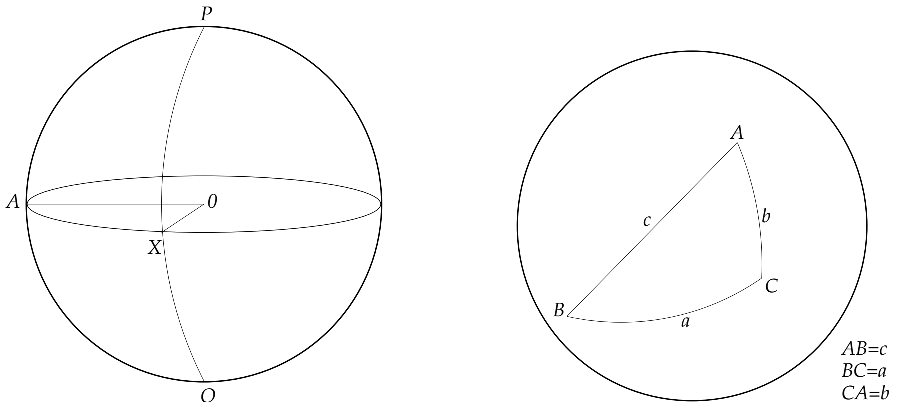
2.1 Some definitions

A sphere is a finite, but unbounded 2-dimensional surface. It is the locus of points that are equidistant from a fixed point (the centre) in 3 dimensions.

Spherical Geometry is the geometry performed on the surface of a sphere. This is a 2-dimensional geometry, but differs significantly from the ordinary 2D plane geometry of Euclid. For example, straight lines do not exist on the surface of a sphere. The equivalent *curves* are the Great Circles.

Great Circles: Any plane passing through the centre of a sphere intersects that sphere in a Great Circle. These objects take on the role of **Geodesics**, linking 2 points on the surface by the shortest path, just like straight lines in Euclidean Geometry.

Spherical Angles: The spherical angle between 2 intersecting great circles is the angle between their plane. This is an important (and slightly subtle) definition to bear in mind.



(a) Sph angle $APX = A\hat{O}X =$ great circle arc AX

(b) The spherical triangle ABC. The sides $a, b, c < \pi$ and so the triangle is confined to a single hemisphere.

Figure 1: For convenience, we shall work throughout with spheres of unit radius.

Spherical Triangles 3 points that lie on the same great circle are analogous to 3 colinear points in plane geometry. However, 3 general points A, B, C that do not lie on the same great circle, may be joined together in pairs by great circle arcs, AB, BC, CA, each of which is $< \pi$ in length. The resulting figure is a Spherical Triangle. It has 3 sides (each of which is the arc of a great circle) and 3 spherical angles (the angles between those great circles' planes). By convention, we denote each angle by a CAPITAL letter, and each side by the small letter corresponding to the opposite angle.

Spherical triangles share many properties with plane ones.

- They are made up of geodesic sections (shortest path between points).
- Any side has a length smaller than the sum of the other two.

But there are important differences too:

- A plane triangles angles sum to $180°$. A spherical triangles angles sum to $> 180°$.
- A spherical triangle may have up to 3 right angles. A plane triangle may have one at the most.

Consider a plane passing through the points ABC, and note that this does not pass through the centre of the sphere. However, we can constrauct a second plane, parallel to the plane ABC that does pass through the sphere's centre. This divides the sphere into 2 hemispheres. Our triangle is confined to one of these, as every part of the spherical triangle is constrained to lie between 0 and π. On this range, cos is single valued, but sin is not. Therefore any angle we wish to deduce should be obtained in terms of its cosine in order to avoid any possible confusion.

Small Circles A plane that does notpass through the centre of a sphere intersects that sphere (if at all) in a small circle. The poles of the small circle are the extremities of the diameter of the sphere that is perpendicular to the small circle's plane (P, Q in figure 2). So a small circle is, quite simply, a circle whose radius is smaller than that of the sphere. In figure 2, the small circle AB is shown, along with its parallel great circle CD. Both circles share P and Q as their poles. Suppose that the small circle has radius r, and let the great circle arc $AP = \theta$. By considering the plane triangle AOS we immediately see that

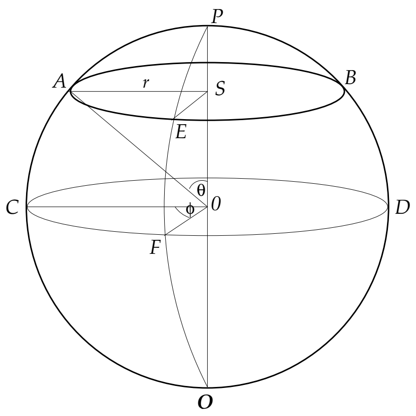


Figure 2: Small circle AB forms a parallel of latitude; it is the locus of points that are a fixed distance from the pole P. The small circle is therefore the analogy of the circle in plane geometry. θ is the separation of the points of the small circle from the pole (cf co-latitude). ϕ is the spherical angle at the pole subtended by the arc in question (cf longitude).

$$AS = A0 \sin A\hat{0}S$$

$$\Rightarrow r = \sin\theta$$

Now consider any point E on the small circle AB. Join the great circle arc PE, and extend it to meet the great circle (equator) CD at F. Clearly the arc PE is equal to θ (see figure 2). We now let the spherical angle $APE = \phi$. Equivalently, $C\hat{0}F = \phi$. Since AS is parallel to

$C0$, and ES to $F0$, it immediately follows that $A\hat{S}E = \phi$. We can now derive the length of the small circle arc AE:

$$AE = rA\hat{S}E = \phi \sin\theta$$

Note: This distance is not the shortest path on the sphere from A to E. The length of the great circle AE will be shorter. It is essential to realise that a small circle **cannot** be part of a spherical triangle.

2.2 Spherical Polar Coordinates

Consider a point A on a sphere, with cartesian coordinates (x, y, z). We know that $x^2 + y^2 + z^2 = 1$. i.e. there exists a degeneracy (1 coordinate is redundant). It is often more convenient to adopt spherical polars (r, θ, ϕ). We have (see figure 4):

- **Radial** coordinate, r - drop this for points on the surface of the sphere.
- **Polar** coordinate, $\theta = Z\hat{0}A$
- **Azimuthal** coordinate, $\phi =$ angle between $Z0A$plane and $z - x$ plane.

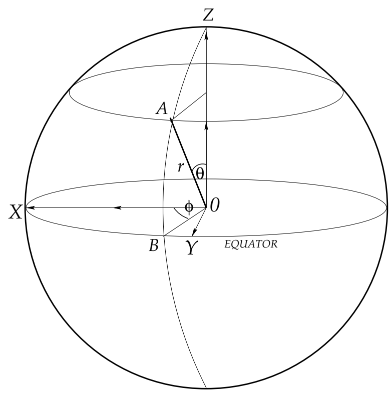


Figure 3: Spherical Polars and Cartesians. X, Y, Z indicate the cartesian axes. In spherical polars, these coordinates are replaced by (r, θ, ϕ), or just (θ, ϕ) for any point on the surface of the sphere. By considering the projection of the vertices of plane triangle $A0B$ onto the cartesian axes, is trivial to find the relationships between these 2 coordinate systems (equation 1)

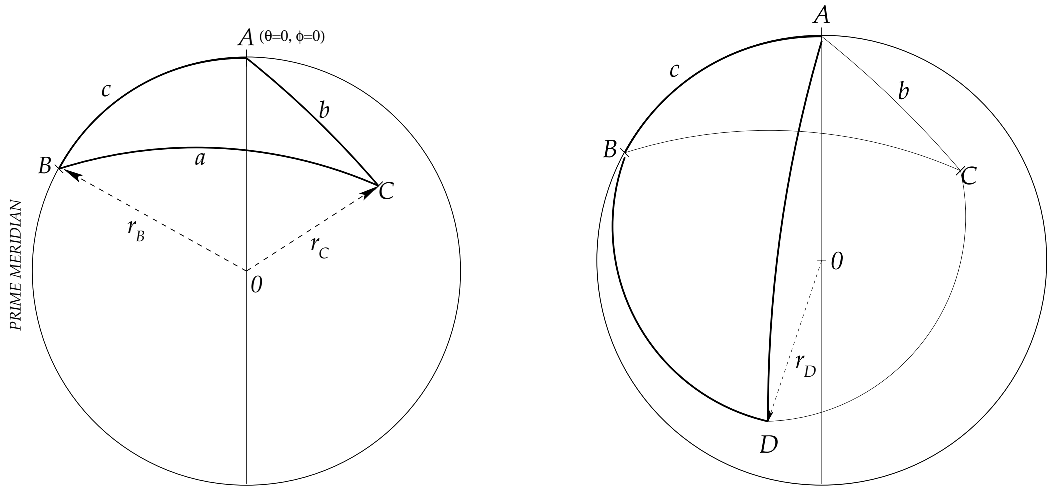
$$x = \sin\theta\cos\phi; \quad y = \sin\theta\sin\phi; \quad z = \cos\theta \tag{1}$$

Any problem in spherical astronomy may be treated by the methods of spherical trig, or by a 3-D vector approach. Which one is adopted is largely a matter of preference.

2.3 Spherical Trig - Basic Results

Here we shall derive the 4 basic equations of spherical trigonometry from first principles. Although not hard, these 4 equations provide a solid and powerful toolkit capable of dealing with any problem in spherical geometry. We shall consider throughout, a spherical triangle ABC on the surface of a sphere of unit radius, and shall adopt the polar coordinate system, (θ, ϕ) (see figure 5).

$AB =$ arc of the reference great circle $(\phi = 0)$.
Therefore B is the point $(\theta = c, \phi = 0)$.



(a) Spherical triangle ABC.

(b) Spherical triangle ABD. Point D placed 90° from both B and C.

Figure 4: We shall use the spherical triangle ABC to derive all of the key equations of spherical trignometry. We set it up so that A is the pole ($\theta = 0, \phi = 0$), and the arc AB lies along the prime meridian (ie the plane $\phi = 0$.

C is the point ($\theta = b, \phi = A$).

We now consider the **vector** positions of the points B and C with respect to the centre of the sphere (and origin of the coordinate frame).

$$\mathbf{r_B} = (\sin\theta_B \cos\phi_B, \sin\theta_B \sin\phi_B, \cos\theta_B) = (\sin c, 0, \cos c)$$

$$\mathbf{r_C} = (\sin b \cos A, \sin b \sin A, \cos b)$$

The angle **between** these 2 vectors is equal to the side BC of the spherical triangle (recall that spherical triangles are made up of great circle arcs).

2.3.1 The Scalar Product and the Cosine Rule

Noting that both $\mathbf{r_B}$ and $\mathbf{r_C}$ are **unit** vectors, consider their scalr product, which is simply the projection of one in the direction of the other:

$$\mathbf{r_B}.\mathbf{r_C} = \cos a \tag{2}$$

Evaluating, we find that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{3}$$

which is the most fundamental formula of spherical trig, and referred to simply as the "Cosine formula".

All formulae of spherical trig may be derived by succesive applications of the cosine formula and a little subsequent algebraic or trigonometric manipulation.

However, 2 further highly useful formulae are directly obtainable by considering the **vector product**, $\mathbf{r_C} \times \mathbf{r_B}$

2.3.2 The Vector Product and the Sine and Analogue Rules

Once again, we first note that the angle between $\mathbf{r_B}$ and $\mathbf{r_C}$ is simply the arc $BC = a$, and so $\mathbf{r_C} \times \mathbf{r_B} = \mathbf{r_D} \sin a$, where $\mathbf{r_D}$ is a unit vector directed toward a point on the sphere that is 90° from both B and C. We call this point D (see overhead and copy onto figure 5), and it has the coordinates $(\theta_D = AD, \phi_D = BAD)$. Noting the relationship of equation 1, we can write this in cartesians as

$$\mathbf{r_D} = (\sin AD \cos BAD, \sin AD, \cos AD)$$

$$\Rightarrow \mathbf{r_C} \times \mathbf{r_B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin b \cos A & \sin b \sin A & \cos b \\ \sin c & 0 & \cos c \end{vmatrix} = \begin{pmatrix} \sin b \cos c \sin A \\ \cos b \sin c - \sin b \cos c \cos A \\ -\sin b \sin c \sin A \end{pmatrix} = \mathbf{r_D} \sin a$$

$$\Rightarrow \begin{pmatrix} \sin b \cos c \sin A \\ \cos b \sin c - \sin b \cos c \cos A \\ -\sin b \sin c \sin A \end{pmatrix} = \sin a \begin{pmatrix} \sin AD \cos BAD \\ \sin AD \sin BAD \\ \cos AD \end{pmatrix} \tag{4}$$

Equating components of equation 4, we can derive 2 important new formulae.

Consider the triangle BAD (which you should by now have copied onto figure 4, but haven't cause you were doodling, so I've drawn it again in figure 5).

- D is the pole of the great circle BC.
- Arc $BD = CD = 90°$.
- BD and CD are perpendicular.
- $ABD = 90° + B$

Apply the Cosine rule (equation 3) to triangle BAD:

$$\cos AD = \sin c \cos(90° + B) = -\sin B \sin c$$

substituting this into the z-component of r_D in equation 4, and equating gives us:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{5}$$

and is known as the "Sine formula". If we now apply *this* to the same triangle BAD:

$$\frac{\sin AD}{\sin ABD} = \frac{\sin AB}{\sin ADB} = \frac{\sin BD}{\sin BAD}$$

we see that

$$\sin AD \sin BAD = \sin ABD = \sin(90° + B) = \cos B$$

which we substitute into the y-component of r_D in equation 4, and equate:

$$\cos b \sin c - \sin b \cos c \cos A = \sin a \sin AD \sin BAD = \sin a \cos B$$

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$

This is known as the "Analogue formula", and is geneerally used in similar situations to the sine formula. There are 6 cases of the analogue formula for a given triangle - see figure 7a.

Inspection of the x-component of the vector product (equation 4) just gives rise to a reformulation of the sine rule, so that's the dot and cross products considered, and three equations derived. The final tool of our spherical toolkit emerges when we consider the sine and cosine rules together.

...see working on board....

$$\cos a \cos C = \sin a \cot b - \sin C \cot B \tag{6}$$

which is known as the "4-parts formula", simply because it relates 4 adjacent "parts" of the triangle (considering either a side or an angle as a "part") - see figure 7b.

2.4 Summary

2.4.1 Cosine formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

2.4.2 Sine formula

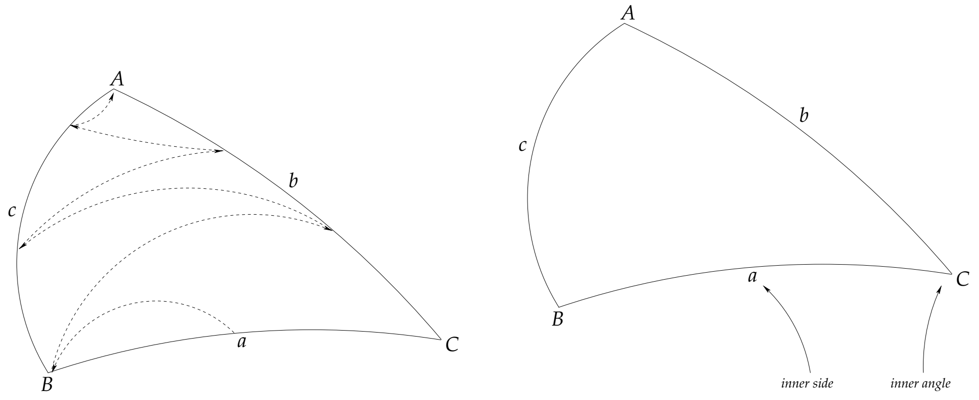
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

2.4.3 Analogue formula

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$

2.4.4 4-parts formula

$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$



(a) The order in which the 6 parts of the triangle appear in the analogue formula ($aBbcbcA$).

(b) The 4-parts formula relates 4 adjacent parts of the spherical triangle ABC. For example B, a, C, b as shown.

Figure 5: Visual aids for the analogue and 4-parts rules. Note that there are 6 permutations of each for any given spherical triangle.

2.5 Example Problem: separation of 2 points on earth

As a first approximation, we shall model the earth as a sphere that is spinning once per day about a fixed rotational axis which intersects the surface at the North and South poles. The great circle with N and S as its poles is known as the **equator**. Any semi-great-circle perpendicular to the equator and terminating at N and S is a **meridian**. We arbitrarily choose to adopt $NGKS$ as our **prime meridian**. Small circles of fixed latitude (hence having N and S as poles) form **parallels**. The parallels and meridians provide a coordinate grid on the surface of this sphere The position of any point, X, is defined by the great circle arc NX and the spherical angle GNX.

latitude $= \phi = 90° - NX$

longitude $= \lambda = GNX$

What is the distance between 2 points, $X(\phi, \lambda)$, and $Y(\phi', \lambda')$ on the earth's surface? Obviously the shortest distance is the **great circle arc** XY, which is a part of the spherical triangle NXY (figure 6 - see board and OHP).

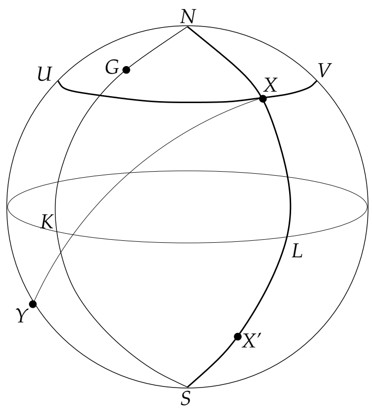


Figure 6: Shortest path between points X and Y is along great circle arc XY.

The e cosine formula (equation 3) is applied to deduce the distance XY in terms of its cosine:

$$\cos XY = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda - \lambda')$$

3 Spherical Astronomy-The Celestial Sphere

We do not have time here to discuss in detail the various coordinate systems used in astronomy. As astronomers you will be familiar at least with the concept of RA and dec.

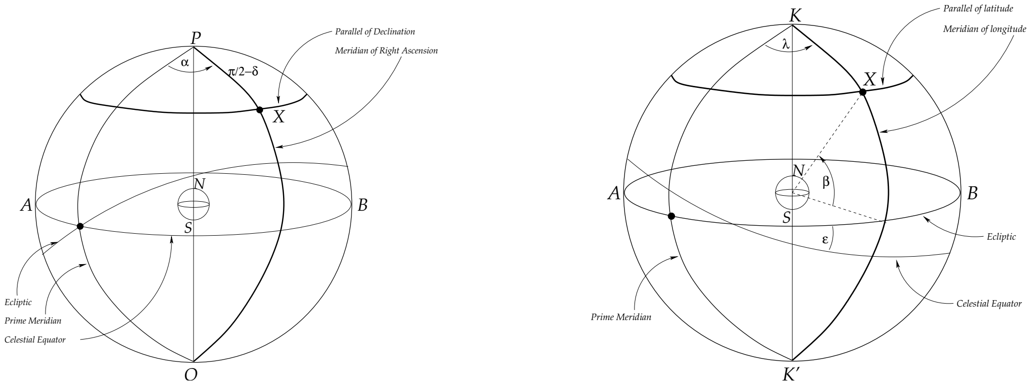
Only brief - not in scope to discuss all coord systems.

3.1 Equatorial Coordinates

- "Latitude" is **Declination**, δ
- "Longitude" is **Right Ascension** α
- "Prime Meridian" is PYQ.

3.2 Ecliptic Coordinates

- "Latitude" is **Ecliptic Latitude**, β
- "Longitude" is **Ecliptic Longitude** λ
- "Prime Meridian" is KYK'.
- $\epsilon \approx 23°27'$ is the obliquity of the ecliptic



(a) The equatorial coordinate system (α, δ) is defined completely by the plane of the ecliptic and the equatorial plane (or equivalently the poles of those planes).

(b) The Ecliptic coordinate system (λ, β)

Figure 7: 2 common astronomical coordinate systems

4 Application 1: Navigation

Before the invention of the marine chronometer (an accurate way of keeping time at sea), a navigator's only way of finding longitude while at sea, was to use the motions of the heavenly bodies as a clock. An ingenious method was devised, and ultimately put into practice by the astronomer royal Neville Maskelyne. This took the known position of the moon, relative to certain bright fixed background stars. These were tabulated at the Royal Observatory in Greenwich. A navigator could then take those tables, and, in principle, determine the time at Greenwich from the measured “Lunar Distance”.

It is beyond this scope of this talk to go into the intricacies of the technique, but I thought I would show you the behind-the-scenes spherical trig that make the technique work.

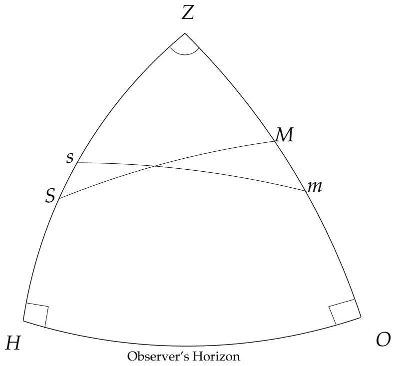


Figure 8: The spherical triangles involved in the calculaton of a lunar distance. The observers horizon is shown along with the true (M, S) and apparent (m, s) positions of the moon and sun respectively. The sun is affected purely by refraction, but in the case iof the moon, one has to take into account the lunar parallax also. Z is the observers zenith.

The position of the moon and sun, as observed from a ship, are affected by atmospheric refraction. In addition, due to its close proximity, there is a parallax effect upon the moons position, and some prior knowledge about one's approximate location must be enterred in order to account for this.

Assuming one can solve the problems of refraction, and lunar parallax, and also assuming that one can actually see both sun and moon, and measure their **altiudes** and the distance between them, then we have the situation illustrated in figure 9. This is now a simple matter of simultneously solving 2 spherical triangles (see OHP).

5 Application 2: Precession and Nutation

Here we introduce DYNAMICS into the framework of spherical trig. In this case we are going to use the Newtonian model of gravitation in order to examine the precession of the earth's orbit. This is of interest to all observational astronomers as it results in the need for EPOCHS as a reference frame for the positions of celestial objects, that we might otherwise think would be fixed in place in the sky.

The equatorial coordinate system (α, δ) is completely defined when the two planes (EQUATOR and ECLIPTIC) are specified, or equivalently, when the North celestial pole, P and the pole of the ecliptic, K are specified.

Let C be the origin of an **inertial reference frame**, which we shall make no attempt to identify or justify here. CP determines the direction of the earth's angular momentum vector. Let us now consider any movement of the points P, K, or X.

Motion of P is Luni-Solar Precession and Nutation

Motion of K is Planetary Precession

Motion of X is Proper Motion

Forces at work: The gravitational pull of the sun, monn, and planets.

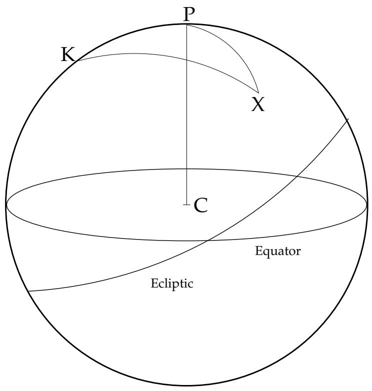


Figure 9: Precession

5.0.1 First Approximation: Zeroth Order

Treat the earth and celestial bodies are perfect spheres. The gravitation between such objects exerts no torque, and so we can treat the earth's angular momentum, **L** as constant. Hence CP is fixed in direction, and P fixed in position on the celestial sphere.

But, largely due to its axial rotation, the earth has in fact developed a slight equatorial oblateness ($\approx 0.3\%$). Therefore the moon and sun both exert a torque which is slowly changing the direction of the earth's angular momentum, **L**, and hence the position of P.

$$\tau \propto \frac{M}{D^3}$$

Now, $\frac{D_{moon}}{D_\odot} = 2.6\times10^{-3}$ and $\frac{M_{moon}}{M_\odot} = 3.7\times10^{-8}$. The moon's effect is therefore approximately twice that of the sun's. In comparison the strongest planetary torques, those of Venus and Jupiter are 5 orders of magnitude weaker, and are not treated here. The resultant torque varies with the configuration of the three bodies, and the resulting motion of P is highly complicated.

We shall describe the secular or long-term change as **luni-solar precession**, the short term effects as **nutation**. This discrimination is arbitrary and made purely for convenience

5.0.2 First Order Perturbation

Treat motion of celestial pole, treating the ecliptic as fixed. We take the **mean torque** exerted by sun and moon as responsible for the luni-solar precession. Changes in this quantity from the mean, are dealt with in terms of nutation, as a higher-order perturbation.