ASTROPHYSICS 3; SEMESTER 1
SOLUTIONS FOR TUTORIAL 5

1. (a) The two processes are extinction by interstellar dust and extinction in the atmosphere. By observing an object at different airmasses/zenith-distances, the extinction of the atmosphere can be determined. The extinction by interstellar dust can be determined e.g. by obtaining the color of the star (i.e. measure magnitudes at different wavelengths) and comparing it with the intrinsic color inferred from the spectral type of the star.

   (b) First it has to be known that an A0 star and the filter systems are defined in such way that it has a zero colour in all filters. Therefore the intrinsic colour of the star is $V-R=0$. Since the observed $V-R$ colour is 0.50, then this must be due to reddening; $A_V - A_R = 0.50$, where A_V is the extinction in magnitudes in the V-band, and A_R in the R-band. We can then calculate the amount of extinction at V from

   $$A_V - A_R = A_V\left(1 - \frac{A_R}{A_V}\right) = 0.50$$

   and

   $$\frac{A_R}{A_V} = \frac{\lambda_V}{\lambda_R} = \frac{5500}{6500},$$

   since it is assumed here that $A_\lambda \propto 1/\lambda$. Therefore $A_V = 3.25$ magnitude.

   (c) M_V=+0.6. Its apparent magnitude m_V, corrected for dust-extinction is $5.0 - 3.25 = 1.75$ magnitude. Therefore this object is 1.15 mag fainter = 2.9 times fainter, than if it were at a distance of 10 parsec. Using the inverse square law, this means that the star is 1.7 times further away = 17 parsec.

   Or, simply from the distance modulus, $m_V - M_V = 5\log D - 5 = 1.15$, we get $D = 17$pc.

2. (a) The star emits $S_*\Delta t$ ionising photons in a time Δt. If in this time the ionised region grows from R to $R+\Delta R$ then the number of newly ionised hydrogen atoms is $4\pi R^2\ \Delta R\ n_H(R)$. Equating the number of ionised hydrogen atoms with the number of ionising photons,

   $$S_*\Delta t = 4\pi R^2\ \Delta R\ n_H(R) = 4\pi R^2\ \Delta R\ n_0(R/R_0)^{-2} = 4\pi R_0^2 n_0\ \Delta R$$

   Hence
   $$\frac{\Delta R}{\Delta t} \rightarrow \frac{dR}{dt} = \frac{S_*}{4\pi R_0^2 n_0}$$

   Integrating,

   $$R = \frac{S_*}{4\pi R_0^2 n_0} t$$

   (b) The rate of recombinations per electron is given by αn_p where α is the radiative recombination coefficient and n_p is the number density of protons. Hence the rate of recombinations per unit volume is

   $$\mathcal{R} = \alpha n_e n_p = \alpha n_H^2$$

In a shell from R to $R + \Delta R$, the recombination rate is

$$\Delta\mathcal{R}(\text{shell}) = 4\pi \mathrm{R}^2 \alpha \mathrm{n}_\mathrm{H}^2 \Delta\mathrm{R} = \frac{4\pi\alpha \mathrm{n}_0^2 \mathrm{R}_0^4}{\mathrm{R}^2}\Delta\mathrm{R}$$

Letting ΔR tend towards zero (becomes dR) and integrating,

$$\mathcal{R} = \int_{R_*}^{R_I} \frac{4\pi\alpha n_0^2 R_0^4}{R^2} dR = 4\pi\alpha n_0^2 R_0^4 \left[\frac{1}{R_*} - \frac{1}{R_I}\right]$$

(c) In the presence of recombinations, some (or all) of the ionising photons are used up re-ionising the hydrogen that has recombined within the HII region, rather than growing the HII region. In this case, in time Δt the number of H-atoms in the newly ionised region equals the number of ionising photons emitted minus the number lost due to recombinations. That is,

$$4\pi R_I^2 n_H(R_I)\, \Delta R_I = S_* \Delta t - 4\pi\alpha n_0^2 R_0^4 \left[\frac{1}{R_*} - \frac{1}{R_I}\right] \Delta t$$

Since $R_I \gg R_*$ we can ignore the $1/R_I$ term. Dividing through by Δt, noting that $n_H(R_I) = n_0(R/R_I)^2$, and going to the limit $\Delta t \rightarrow 0$,

$$4\pi R_0^2 n_0 \frac{dR_I}{dt} = S_* - \frac{4\pi\alpha n_0^2 R_0^4}{R_*}$$

Integrating,

$$R_I = \left(\frac{S_* - (4\pi\alpha n_0^2 R_0^4/R_*)}{4\pi R_0^2 n_0}\right) t$$

$$R_I = \left(\frac{S_*}{4\pi R_0^2 n_0} - \frac{\alpha n_0 R_0^2}{R_*}\right)$$

Putting in the numbers, after 10^4yrs, $R_I = 2.89 \times 10^{18}$m $=$ 88pc.

(d) If the bracketted term in the equation derived in (c) above is negative, then this implies that the size of the ionised region is negative and doesn't grow with t. This is unphysical - so how can this be? It's because the assumption we made of $R_I \gg R_*$ is not valid in this case. This happens at large values of n_0.

Instead here we go back to:

$$4\pi R_I^2 n_H(R_I) \Delta R_I = S_* \Delta t - 4\pi\alpha n_0^2 R_0^4 \left[\frac{1}{R_*} - \frac{1}{R_I}\right] \Delta t$$

In the steady state when equilibrium is reached, $\Delta R_I = 0$: all of the ionising photons are used up balancing recombination, and the HII region stops growing. This means that

$$S_* = 4\pi\alpha n_0^2 R_0^4 \left[\frac{1}{R_*} - \frac{1}{R_I}\right]$$

or re-arranging this,

$$\frac{1}{R_I} = \frac{1}{R_*} - \frac{S_*}{4\pi\alpha n_0^2 R_0^4}$$

Putting the numbers in with $n_0 = 10^9\mathrm{m}^{-3}$, $R_I = 2.02 \times 10^9$m. In other words, the limit of the ionised region is only 2×10^7m above the stellar surface.