ASTROPHYSICS 3; SEMESTER 1
SOLUTIONS FOR TUTORIAL 4

1. (a) Rearranging the mass continuity equation to express this in terms of $M(r)$ and differentiating gives:

$$\frac{dM(r)}{dr} = \frac{-1}{G}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right]$$

Substituting that into the equation of hydrostatic equilibrium then gives

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho$$

(b) $P = K\rho^\gamma$. Substituting that into the above equation gives

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dK\rho^\gamma}{dr}\right] = -4\pi G\rho$$

Now use dimensionless variables:

$$\tilde{r} \equiv r/R; \quad \tilde{\rho} \equiv \rho/\rho_{\rm C};$$

Substituting in for these,

$$\frac{1}{(R\tilde{r})^2}\frac{d}{d(R\tilde{r})}\left[\frac{(R\tilde{r})^2}{(\rho_{\rm C}\tilde{\rho})}\frac{dK(\rho_{\rm C}\tilde{\rho})^\gamma}{d(R\tilde{r})}\right] = -4\pi G(\rho_{\rm C}\tilde{\rho})$$

and hence re-arranging,

$$4\pi\tilde{r}^2\tilde{\rho} = -\left[\frac{K\rho_{\rm C}^{\gamma-2}}{GR^2}\right]\frac{d}{d\tilde{r}}\left(\frac{\tilde{r}^2}{\tilde{\rho}}\frac{d\tilde{\rho}^\gamma}{d\tilde{r}}\right).$$

The constant $K\rho_{\rm C}^{\gamma-2}/GR^2$ in this equation must be dimensionless (as the rest of the equation is) and constant (given that all stars must have the same solution as they have the same boundary conditions of $\tilde{\rho} = 1$ at $\tilde{r} = 0$, $\tilde{\rho} = 0$ at $\tilde{r} = 1$ and $d\tilde{\rho}/d\tilde{r} = 0$ at $\tilde{r} = 0$ (by spherical symmetry, since the pressure gradient must vanish there)). Hence, $\rho_{\rm C}^{\gamma-2}/R^2 = $ constant, and thus $R^2 \propto \rho_{\rm C}^{\gamma-2}$ or $R \propto \rho_{\rm C}^{\gamma/2-1}$.

(c) For non-relativistic white dwarfs, $\gamma = 5/3$ and thus $R \propto \rho_{\rm C}^{-1/6}$, or $\rho_{\rm C} \propto R^{-6}$. Since $M \propto \rho_{\rm C}R^3$ then $M \propto R^{-3}$, or $R \propto M^{-1/3}$.

For relativistic white dwarfs $\gamma = 4/3$, and so similarly $R \propto \rho_{\rm C}^{-1/3}$ and $M \propto \rho_{\rm C}R^3 \propto R^0$. Mass is independent of radius.

2. We discussed and derived the equations of state for degenerate gas based on the consideration of the occupation number and Fermi energy. Here, by looking at these equations of state, we try to derive the conditions for the transitions between the three states, i.e. ideal gas state, non-relativistically degenerate state, and relativistically degenerate state.

   (a) The condition for the transition from an ideal gas state to a non-relativistically degenerate state can be derived by considering the case where the pressures for these two states are the same. Therefore, for a given T, we can derive the density ρ for this transition by equating $K_{\rm nr}\rho^{5/3}$ to $\rho kT/\bar{m}$, where $\bar{m}$ is the mean mass per particle.

   If we evaluate $K_{\rm nr}$ taking μ_e as 1, which is the case for pure ionized hydrogen (though the degenerate core of evolved stars are not made of hydrogen),

$$K_{\rm nr} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1}{\mu_e m_p}\right)^{5/3} = 1.0\times 10^7 \text{ (MKS unit)}.$$

   From

$$K_{\rm nr}\rho^{5/3} = \frac{\rho kT}{\bar{m}},$$

   we obtain

$$\rho = \left(\frac{kT}{K_{\rm nr}\bar{m}}\right)^{3/2} = 6.7\times 10^{-5}T^{3/2} \text{ kg/m}^3.$$

   So, if the density is larger than this, the gas is degenerate.

   Since we have electron number density $n_e = 8\pi p_F^3/(3h^3)$ (taking $g = 2$) and $\epsilon_F = p_F^2/2m_e$, ϵ_F can be written as

$$\epsilon_F = \frac{h^2}{8m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1}{\mu_e m_p}\right)^{2/3}\rho^{2/3} = K_{\rm nr}\frac{20}{8}\mu_e m_p \rho^{2/3}.$$

   Then the condition $\epsilon_F = kT$ converts to

$$\frac{20\mu_e m_p}{8\bar{m}}K_{\rm nr}\rho^{5/3} = \frac{\rho}{\bar{m}}kT,$$

   which is very roughly the same as the above condition since $\bar{m} = 0.5m_p$. (If you calculate the density in terms of temperature assuming $\mu_e = 1$, then you will get $\rho = 5.9\times 10^{-6}T^{3/2}$ kg/m^3.)

   Another way of saying this is that a pressure is of the order of an energy density, so we have $P \sim \epsilon_F\cdot n_e$ for an electron-degenerate gas and $P \sim kT\cdot n_e$ for an ideal gas, and thus equating these two pressures is equivalent to equating ϵ_F and kT.

   (b) We obtain

$$K_{\rm r} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{1}{\mu_e m_p}\right)^{4/3} = 1.2\times 10^{10} \text{ (MKS unit)}.$$

   for $\mu_e = 1$.

   By equating $K_{\rm nr}\rho^{5/3}$ to $K_{\rm r}\rho^{4/3}$, we get the transition density of

$$\rho = \left(\frac{K_{\rm r}}{K_{\rm nr}}\right)^3 = 1.7\times 10^9 \text{ kg/m}^3.$$

Using $\rho = \mu_e m_p n_e$ and $n_e = 8\pi p_F^3/(3h^3)$, the relation $\rho = (K_\mathrm{r}/K_\mathrm{nr})^3$ reduces to $p_F = 5m_e c/4$. (Of course this is much more easily seen if you go back to the original calculation of $P = \int_0^{p_F} \frac{1}{3} pv 8\pi p^2 dp/h^3$.) Thus, $\epsilon_F \sim cp_F \sim m_e c^2$. (If you want, you can calculate it more accurately using $p = \beta\gamma mc$ and $\epsilon = (\gamma - 1)mc^2$ where $\beta = v/c, \gamma = 1/\sqrt{1-\beta^2}$.)

3. From

$$\frac{3}{2} kT \frac{1}{\bar{m}} \simeq \frac{GM}{R},$$

we obtain the temperature T as

$$T \simeq \frac{2\bar{m}GM}{3kR} \simeq 8 \times 10^5 \text{ K}.$$

From the criterion in (2a) above, the density required for degeneracy is

$$\rho = 6.7 \times 10^{-5} \times (8 \times 10^5)^{3/2} = 4.8 \times 10^4 \text{kg/m}^3.$$

On the other hand, the average density of this brown dwarf is

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = 1.4 \times 10^4 \text{ kg/m}^3.$$

We see that this average density is just below the critical density for degeneracy, but for a $P \propto \rho^{5/3}$ polytrope, the central density ρ_c is about $6\bar{\rho} = 8 \times 10^4$ kg/m^3, so the core of the brown dwarf will be degenerate.