

# ASTROPHYSICS 3; SEMESTER 1

## TUTORIAL 4: Degenerate stars

1. The equations of mass continuity and hydrostatic equilibrium are

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \quad \text{and} \quad \frac{1}{\rho} \frac{dP}{dr} = -\frac{GM(r)}{r^2}.$$

- (a) Combine these to show that

$$\frac{1}{r^2} \frac{d}{dr} \left[ \frac{r^2}{\rho} \frac{dP}{dr} \right] = -4\pi G \rho$$

(b) For gas with a polytropic equation of state,  $P = K\rho^\gamma$ , use this to derive a 2nd order differential equation in density and radius alone. Using dimensionless variables or otherwise, show that the term  $K\rho_C^{\gamma-2}/GR^2$  is a dimensionless constant, and hence that  $R \propto \rho_C^{\gamma/2-1}$ .

(c) White dwarfs obey a polytropic equation of state,  $P = K\rho^\gamma$ , where  $\gamma = 5/3$  for non-relativistic polytropes and  $\gamma = 4/3$  for relativistic polytropes. Show that the result in (b) demonstrates that for non-relativistic white dwarfs  $R \propto M^{-1/3}$  whilst for relativistic white dwarfs the mass is independent of radius.

2. The equation of state for an ideal gas is

$$P_{\text{ideal}} = \frac{\rho k_B T}{\bar{m}},$$

where  $\rho$  is the total mass density of the gas,  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$  is the Boltzmann constant, and  $\bar{m} = 0.5m_p$  is the mean mass per particle, where  $m_p = 1.67 \times 10^{-27} \text{ kg}$ .

The equation of state for a fully degenerate non-relativistic gas is

$$P_{\text{deg}} = K_{\text{nr}} \rho^{5/3},$$

where

$$K_{\text{nr}} = \frac{h^2}{20m_e} \left( \frac{3}{\pi} \right)^{2/3} \left( \frac{1}{\mu_e m_p} \right)^{5/3},$$

where  $\mu_e$  is the mean molecular weight per free electron,  $m_e = 9.11 \times 10^{-31} \text{ kg}$  is the mass of an electron, and  $h = 6.63 \times 10^{-34} \text{ Js}$  is Planck's constant.

The equation of state for a fully degenerate relativistic gas is

$$P_{\text{deg}} = K_{\text{r}} \rho^{4/3},$$

where

$$K_r = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \left(\frac{1}{\mu_e m_p}\right)^{4/3},$$

where  $c = 2.998 \times 10^8 \text{ ms}^{-1}$  is the speed of light.

(a) Evaluate  $K_{nr}$ . By comparing the equation of state for an ideal gas with that of a degenerate non-relativistic gas, estimate the minimum mass density  $\rho$  required for a gas of temperature  $T$  to be degenerate. Show that this condition is roughly (within  $\sim 1$  order of magnitude) equivalent to the condition  $\epsilon_F \sim kT$ , where  $\epsilon_F$  is the Fermi energy for the electrons.

(b) Evaluate  $K_r$ . By comparing the equation of state for a degenerate non-relativistic gas with that of a degenerate relativistic gas, estimate the minimum mass density required for the gas to be relativistically degenerate. Show that this condition is roughly equivalent to the condition  $\epsilon_F \sim m_e c^2$ .

3. Brown dwarfs are stars that have masses and central temperatures just below the requirements of hydrogen thermonuclear fusion. They are visible shortly after they form, but as they cool they grow dim until they become invisible. It is possible that brown dwarfs comprise a large fraction of the mass in the Galaxy.

Consider a brown dwarf of mass  $M = 0.01M_\odot$  and radius  $R = 0.1R_\odot$ , where  $M_\odot = 2 \times 10^{30} \text{ kg}$  is a solar mass and  $R_\odot = 7 \times 10^8 \text{ m}$  is a solar radius. Estimate the internal temperature of the brown dwarf by equating the thermal energy per unit mass ( $\frac{3}{2}k_B T / \bar{m}$ ) to the gravitational energy per unit mass of the star. Is the interior of the brown dwarf (non-relativistically) degenerate, based on the criterion obtained in 2(a) above? (For a polytrope satisfying  $P \propto \rho^{5/3}$ , the central density is 6 times the average density of the star.)

### Hand-in Question; Physics of Stars & Nebulae, 2003 exam

B2 Suppose that a protostar is in a gravitational contraction phase, maintaining hydrostatic equilibrium described as  $dP/dr = -\rho GM(r)/r^2$ .

(a) Using either dimensionless variables or the Virial Theorem, and assuming the ideal gas law, show that the central temperature  $T_c$  of a proto-star scales with the mass and central density as  $T_c \propto M^{2/3} \rho_c^{1/3}$ . [5]

(b) As the contraction proceeds and the density gets higher, electrons can become degenerate and the equation of state can change. Degeneracy sets in when  $\Delta x \Delta p \simeq \hbar$ , where  $\Delta x$  is the mean distance between electrons and  $\Delta p$  is the dispersion of electron momentum. Write  $\Delta x$  in terms of electron density  $n_e$ . Write  $\Delta p$  in terms of electron mass  $m_e$  and temperature  $T$ , based on equipartition (or Boltzmann distribution). Then show that this boundary for degeneracy corresponds to  $T \propto \rho^{2/3}$ . [6]

(c) Assume that the gravitational contraction continues to the stage where degeneracy sets in. Based on these two relations between  $T_c$  and  $\rho_c$ , sketch the track of a collapsing proto-star of given mass  $M$  on the  $\log T_c - \log \rho_c$  plane, and indicate how this differs for stars of higher and lower mass. Explain that there is an upper limit on the temperature  $T_{\max}$  which can be achieved by the gravitational contraction, and show that  $T_{\max} \propto M^{4/3}$ . [6]

(d) If the mass of the protostar is very small, will it become a main-sequence star? Why (or why not)? [3]