ASTROPHYSICS 3; SEMESTER 1
SOLUTIONS FOR TUTORIAL 3

1. (a) Define

$$g(E) = \ln p(E) = -\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}.$$

Then

$$g'(E) = dg/dE = -\frac{1}{kT} + \frac{1}{2}\frac{E_G^{1/2}}{E^{3/2}}.$$

Setting $g'(E) = 0$ gives

$$E_0 = \left(\frac{E_G(kT)^2}{4}\right)^{1/3}.$$

(b) At this energy,

$$g(E_0) = -\frac{E_0}{kT} - \left(\frac{E_G}{E_0}\right)^{1/2} = -\left(\frac{E_G}{4kT}\right)^{1/3} - \left(\frac{2E_G}{kT}\right)^{1/3} = -3\left(\frac{E_G}{4kT}\right)^{1/3}$$

Differentiating $g(E)$ again,

$$g''(E) = \frac{d^2g}{dE^2} = -\frac{3}{4}\frac{E_G^{1/2}}{E^{5/2}},$$

so at the peak,

$$g''(E_0) = -\frac{3}{4}\frac{E_G^{1/2}}{E_0^{5/2}} = -\frac{3}{4}\left(\frac{E_G}{E_0^3}\right)^{1/2}\frac{1}{E_0} = \frac{3}{2kT}\left(\frac{4}{E_G(kT)^2}\right)^{1/3}.$$

Taylor-expanding $g(E)$ about the peak to the second order, and noting that $g'(E_0) = 0$, we have the following:

$$g(E) \simeq g(E_0) + \frac{1}{2}\left(\frac{d^2g}{dE^2}\right)_{E=E_0}(E - E_0)^2.$$

Hence

$$g(E) \simeq -3\left(\frac{E_G}{4kT}\right)^{1/3} - \frac{1}{2}\frac{3}{2kT}\left(\frac{4}{E_G(kT)^2}\right)^{1/3}(E - E_0)^2$$

(c) Then,

$$p(E) = e^{g(E)} \simeq e^{-3(E_G/4kT)^{1/3}}e^{-(E-E_0)^2(3/4kT)(4/E_G(kT)^2)^{1/3}},$$

and equating this with $p(E) \propto e^{-(E-E_0)^2/(\Delta/2)^2}$ means that

$$(\Delta/2) = (4kT/3)^{1/2}(E_G(kT)^2/4)^{1/6}$$

and thus

$$\Delta = \frac{4}{3^{1/2}2^{1/3}}E_G^{1/6}(kT)^{5/6}.$$

(d) $R_{AB} \propto T^{-3/2} p(E_0) \Delta$

$$p(E_0) = e^{g(E_0)} \simeq e^{-3(E_G/4kT)^{1/3}}$$

and hence

$$R_{AB} \propto T^{-3/2} T^{5/6} e^{-3(E_G/4kT)^{1/3}}$$

so

$$\ln R_{AB} = -\frac{2}{3} \ln T - 3 \left(\frac{E_G}{4kT} \right)^{1/3} + \text{ const.}$$

Therefore,

$$\frac{d \ln R_{AB}}{d \ln T} = \frac{2}{3} + \left(\frac{E_G}{4kT} \right)^{1/3},$$

noting (when calculating the second term) that

$$\frac{d}{d \ln T} = T \frac{d}{dT}.$$

(e) pp-chain starts from $p + p \rightarrow d + e^+ \nu_e$. So $Z_A = Z_B = 1$, $m_{\text{red}} = m_H/2$. Then we have $E_G = (\pi\alpha)^2 m_H c^2 = 7.9 \times 10^{-14}$J. Then $d \ln R_{AB}/d \ln T \simeq 3.5$ (so $R_{AB} \propto T^{3.5}$) for $T \sim 2 \times 10^7$K.

(f) CNO-cycle starts from $p + ^{12}\text{C} \rightarrow ^{13}\text{N} + \gamma$. So $Z_A = 1$ and $Z_B = 6$, $m_{\text{red}} = (m_H \cdot 12 m_H)/(m_H + 12 m_H) = (12/13) m_H$. Then we have $E_G = (6\pi\alpha)^2 2 (12/13) m_H c^2 = 5.2 \times 10^{-12}$J. Then $d \ln R_{AB}/d \ln T \simeq 16$ (so $R_{AB} \propto T^{16}$) for $T \sim 2 \times 10^7$K.

2. (a) Consider a shell of gas at radius r, thickness dr. The volume of the shell is $4\pi r^2 dr$, so the mass of gas contained within it is $dm = 4\pi r^2 \rho dr$. The gravitational potential energy of the shell is then $-GM(r)dm/r$, where $M(r)$ is the mass contained within radius r, and hence it is $-(GM(r)/r) 4\pi r^2 \rho dr$. Integrating across all such shells from $r = 0$ to $r = R$ then gives the total gravitational potential energy of the star.

(b) For uniform density, $\rho = M/(\frac{4}{3}\pi R^3)$ at all radii, and $M(r) = \frac{4}{3}\pi r^3 \rho$. Hence

$$\Omega = \int_0^R -\frac{G \frac{4}{3} \pi r^3 \rho}{r} 4\pi r^2 \rho dr$$

$$\Omega = -\frac{16\pi^2 G \rho^2}{3} \int_0^R r^4 dr$$

$$\Omega = -\frac{3GM^2}{5R} \propto \frac{1}{R}$$

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(c) In a shell of the star from radius r to $r + dr$ the total number of particles is $dN = 4\pi r^2 n dr$ where n is the particle density. The kinetic energy of each particle is 3kT/2 (equipartition), and hence the total kinetic energy in the shell is $dU = (3kT/2) 4\pi r^2 n dr$. Integrating,

$$U = \int_0^R \frac{3}{2} n k T 4\pi r^2 dr.$$

From the ideal gas law, $P = nKT$, and hence

$$U = \int_0^R \frac{3}{2} P \cdot 4\pi r^2 dr.$$

(d) The equation of hydrostatic equilibrium says that

$$-\frac{1}{\rho}\frac{dP}{dr} = \frac{GM(r)}{r^2}.$$

Substituting this into (a),

$$\Omega = \int_0^R -\frac{dP}{dr} 4\pi r^3 dr.$$

Integrating by parts:

$$\Omega = \left[P \cdot 4\pi r^3\right]_0^R - 3\int_0^R P \cdot 4\pi r^2 dr.$$

The first term is zero. The second term gives $\Omega = -2U$. This is the Virial Theorem.