ASTROPHYSICS 3; SEMESTER 1
TUTORIAL 3: Nuclear Fusion and Hydrostatic Equilibrium

1. The rate R_{AB} at which nuclei of atomic numbers Z_A and Z_B will fuse by quantum tunnelling depends on temperature according to

$$R_{AB} \propto T^{-3/2} \int_0^\infty S(E)p(E)dE,$$

where $S(E)$ is a slowly varying function of the kinetic energy of the nuclei, and $p(E)$ describes the probability for fusion, balancing the Boltzmann distribution of particle energies against the probability for tunnelling. (Nuclei with higher kinetic energies fuse more easily, but according to the Boltzmann factor, there are fewer of them.) The coefficient $T^{-3/2}$ comes from the Maxwellian distribution of particle speeds, and accounts for the relative speed of the colliding nuclei (reaction rate is $\propto n\,\sigma\,v$), where σ is the fusion cross-section. The principal temperature behaviour of $p(E)$ is given by

$$p(E) = \exp\left[-\frac{E}{k_{\rm B}T} - \left(\frac{E_G}{E}\right)^{1/2}\right],$$

where E_G is the Gamow energy $E_G = (\pi\alpha Z_A Z_B)^2 2m_{\rm red}c^2$, $\alpha \simeq 1/137$ is the fine–structure constant, and $m_{\rm red} = m_A m_B/(m_A + m_B)$ is the reduced mass of the nuclei.

(a) Show that the energy E_0 that maximizes $p(E)$ is given by $E_0 = \left(\frac{E_G(kT)^2}{4}\right)^{1/3}$ [Hint: this will be the same as the energy that maximizes $g(E) = \ln p(E)$ – why?]

(b) Expand $g(E)$ about this maximum energy E_0 to second order using a Taylor series [$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a)/2 + \ldots$], to show that

$$g(E) \simeq -3\left(\frac{E_G}{4kT}\right)^{1/3} - \frac{1}{2}\frac{3}{2kT}\left(\frac{4}{E_G(kT)^2}\right)^{1/3}(E-E_0)^2$$

(c) Use these results to estimate the dependence of R_{AB} on temperature T by approximating $p(E)$ as a Gaussian (or Normal) distribution of the form

$$p(E) \propto \exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right],$$

where E_0 is the energy E at which $p(E)$ is a maximum, and Δ is the effective width of $p(E)$.

(d) Since $p(E)$ is sharply peaked about $E = E_0$ with a width Δ, a good approximation for the fusion rate is $R_{AB} \propto T^{-3/2}p(E_0)\Delta$. Using this, show that

$$\frac{d\ln R_{AB}}{d\ln T} \simeq -\frac{2}{3} + \left(\frac{E_G}{4k_{\rm B}T}\right)^{1/3}.$$

(e) Show that $E_G = 8 \times 10^{-14}$ J for the pp chain. Use this with the result from part (d) to show that for temperatures T near 2×10^7 K, $R_{AB} \propto T^{3.5}$.

(f) Show that $E_G = 5 \times 10^{-12}$ J for the CNO cycle. Use this with the result from part (d) to show that for temperatures T near 2×10^7 K, $R_{AB} \propto T^{16}$.

2. A star is in a hydrostatic equilibrium.

(a) Explain why the gravitational potential energy of the star is given by

$$\Omega = \int_0^R -\frac{GM(r)}{r} 4\pi r^2 \rho dr,$$

where R is the radius of the star, ρ is a mass density, and $M(r)$ is the mass within radius r.

(b) If the density is uniform, derive Ω in terms of the mass M and radius R of the star. Show that for a given M, Ω is proportional to $1/R$.

(c) Show that the total thermal energy U of the star is given by

$$U = \int_0^R \frac{3}{2} P \cdot 4\pi r^2 dr,$$

using the ideal gas law and assuming that the average thermal energy of a particle is $3kT/2$.

(d) Derive the relation between U and Ω. Explain the physical meaning of the relation.

Hand-in Question; modified from Astro-3 exams 2008 B1.3 and 2010 B1

(a) The equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}.$$

Explain what is meant by this equation, and define the terms involved. Using dimensionless variables, show that the hydrostatic equilibrium equation implies that the central temperature of a star scales as $T_c \propto M/R$. **[5]**

(b) What are meant by *the Virial Theorem* and *Equipartition of Energy*? Demonstrate that these relations lead to the same proportionality found in part (a). Using the Virial Theorem or otherwise, derive a rough estimate of the internal temperature of the Sun. **[5]**

(c) Energy is generated in the centre of the Sun through hydrogen fusion. Estimate the gas temperature required for two protons to have sufficient energy to overcome their Coulomb repulsion (you can assume that the proton radius is $r_p = 10^{-15}$m). Why does this temperature differ from that derived above? Explain why the protons undergoing fusion all have energies in a very narrow energy range. **[6]**

(d) Briefly describe the process by which photons produced by fusion reactions at the centre of a star escape from the star. Considering both this and the Virial Theorem, explain why the sun does not explode like a nuclear bomb, but rather burns in a controlled manner for 9 billion years. **[4]**