ASTROPHYSICS 3; SEMESTER 1
SOLUTIONS FOR TUTORIAL 2: Various Estimates

(a)

$$\bar{\rho} = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = 1410 \text{ kg m}^{-3}.$$

From the equation of hydrostatic equilibrium,

$$-\frac{P_c}{\rho_c R_\odot} \simeq -\frac{GM_\odot}{R_\odot^2}.$$

The equation of state of the gas at the centre of the sun gives $P_c = \rho_c kT_c/\bar{m}$, where $\bar{m} = m_H/2$ for fully ionized hydrogen. Therefore,

$$T_c \simeq \frac{Gm_H M_\odot}{2kR_\odot} = 1.2 \times 10^7 K.$$

(b) Roughly, the typical collapse speed is $\dot{r} \sim \sqrt{\frac{GM}{R}}$ (kinetic energy is roughly equal to potential energy). Therefore typical collapse time τ is $\tau \simeq R/\dot{r} \sim \sqrt{R^3/GM} \sim 1/\sqrt{G\rho}$.

More accurately, energy conservation says

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = -\frac{GM}{R},$$

at radius r, where M is the total mass of the star. Therefore,

$$\frac{dr}{dt} = \sqrt{\frac{2GM}{r} - \frac{2GM}{R}}$$

and the collapse time τ is

$$\tau = \int_0^R \frac{dr}{\sqrt{\frac{2GM}{r} - \frac{2GM}{R}}} = \frac{R}{\sqrt{2GM/R}} \int_0^1 \frac{dy}{\sqrt{1/y - 1}}$$

where $y = r/R$. Substituting y as $\sin^2\theta$, we get

$$\tau = \frac{R}{\sqrt{2GM/R}}\frac{\pi}{2} = \sqrt{\frac{\pi^2 R^3}{8GM}} = \sqrt{\frac{3\pi}{32}\frac{1}{G\rho}} \simeq 30\text{minutes}.$$

This timescale is called a free-fall time, and it is independent of radius, and it depends only on ρ.

(c) For a blackbody, $L_\odot = 4\pi R_\odot^2 \sigma_{SB} T_{\text{eff}}^4$. Therefore, $T_{\text{eff}} = 5770\text{K}$. Differentiating the Planck function and setting it to zero to find the peak wavelength gives Wien's displacement law:

$$\lambda_{\text{max}} T = 2900\mu\text{m K}.$$

When $T = 5770\text{K}$, $\lambda_{\text{max}} = 0.50\mu\text{m}$. This specific wavelength corresponds to yellowish green color.

(d) $R_{SG} = \sqrt{\frac{L_{SG}}{4\pi\sigma_{SB}T_{\text{eff}}^4}} = 1.8 \times 10^{10} m = 25 R_\odot.$

(e) If the mean density is the same as the Sun, $M = \bar{\rho}_{\odot}\frac{4}{3}\pi R_{SG}^3 = 1.6 \times 10^4 M_{\odot}$. This is much too big. If the mass is only $25 M_{\odot}$ then for the mean density we get

$$\bar{\rho}_{SG} = \frac{M_{SG}}{\frac{4}{3}\pi R_{SG}^3} = \frac{M_{SG}}{M_{\odot}}\left(\frac{R_{\odot}}{R_{SG}}\right)^3 \bar{\rho}_{\odot} = 0.0016\bar{\rho}_{\odot} = 2.2\text{kg m}^{-3}.$$

(f) If the photon travels without scattering, it takes $R_{SG}/c \simeq 1$ minute.

(g) Mean free path l is written as $l = 1/n_s\sigma_s$, where n_s is the number density of scatterers of cross-section σ_s.
Because the gas in the supergiant is almost completely ionized, the photons will scatter off the electrons, with a mean free path of $l = 1/n_e\sigma_T$, where n_e is electron number density and σ_T is Thomson scattering cross section.
For $\bar{\rho}_{SG} = 2.2\text{kg m}^{-3}$,

$$n_e \simeq n_{proton} = \bar{\rho}_{SG}/m_H = 1.4 \times 10^{27} m^{-3}.$$

Therefore $l = 1/n_e\sigma_T = 11\text{m}$.

(h) If a specific kind of particles, say photons, are concentrated at some location within a cloud of other particles, these photons will diffuse by a random walk through collisions with other particles. Then, on average, the photon proceeds the distance of $\sqrt{N} \cdot l$ after N scatterings. Another way of saying this is that to proceed a distance D, a photon has to go through scatterings of $(D/l)^2$ times.
For the case of the Supergiant here, the number of scatterings needed to escape is $N = (R_{SG}/l)^2 = 2.5 \times 10^{18}$. Thus, the escape time for the photon (to diffuse from the center of the star to the surface) is

$$t = N\frac{l}{c} = 9 \times 10^{10} sec = 2900 yrs,$$

noting that the total path length travelled by the photon is $N \cdot l$.

(i) Conservation of energy says that the combined kinetic energy of the protons at large separation (where electrostatic potential energy is zero) must be equal to or greater than the electrostatic potential energy at the peak of the Coulomb barrier (ie. at $r \sim r_{\text{N}}$). Hence, $2 \times \frac{1}{2}m_p v^2 = \frac{e^2}{4\pi\epsilon_0 r_{\text{N}}}$. This gives $v = 1.2 \times 10^7$m/s.
Using equipartition of energy, $\frac{1}{2}m_p v^2 \approx \frac{3}{2}kT$ and hence $T \approx 10^{10}$K.

(j) Differentiating the Maxwell-Boltzmann distribuiton gives

$$\frac{df(v)}{dv} = 4\pi\left(\frac{m}{2\pi kT}\right)^{0.5} \exp\left[\frac{-mv^2}{2kT}\right]\left(2v - \frac{mv^3}{kT}\right).$$

The function peaks when $\frac{df(v)}{dv} = 0$, ie at $v = \sqrt{2kT/m_p} = 4.5 \times 10^5$m/s.
$f(v) \propto v^2 \exp(-mv^2/2kT)$. Hence

$$\frac{f(v_{\text{peak}})}{f(v_{\text{fusion}})} = \left(\frac{v_{\text{peak}}}{v_{\text{fusion}}}\right)^2 \exp\left[\frac{-m_p(v_{\text{peak}}^2 - v_{\text{fusion}}^2)}{2kT}\right]$$

This has a value of $\approx \exp(-725)$ which is infinitessimally small.

(k) On fusing 4 protons into one Helium nucleus, energy output is given by $E = (\Delta m)c^2 = (4m_{\text{H}} - m_{\text{He}})c^2 = 4.4 \times 10^{-12} J$. Hence, fusion energy per proton is $1.1 \times 10^{-12} J$. The total number of protons in the sun is roughly $M_{\odot}/m_{\text{H}}$. If the sun burns 10% of these, the total energy output is $E_{\text{tot}} = 0.1 \times (M_{\odot}/m_{\text{H}}) \times 1.1 \times 10^{-12} J \approx 1 \times 10^{44}$J. The lifetime of the sun is then this, divided by the rate at which the energy is radiated ($L_{\odot}$), ie. 3×10^{17}s $\approx 10^{10}$yr.