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Physics of Stars (3): Post Main Sequence Stars

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1 Stellar evolution and the giant branch

We deduced earlier that stars will have a finite lifetime, which is about 9 Gyr for the Sun, and scales very roughly as M^{-3}. After this time, a significant proportion of the initial nuclear fuel in the core will have been fused, and the star can no longer shine on the main sequence. In this course, we will not look at the subsequent evolution in much detail, since it is the subject of a module in Senior Honours. However, we shall look very briefly at the processes that occur. The main events can be seen in Figure 1.

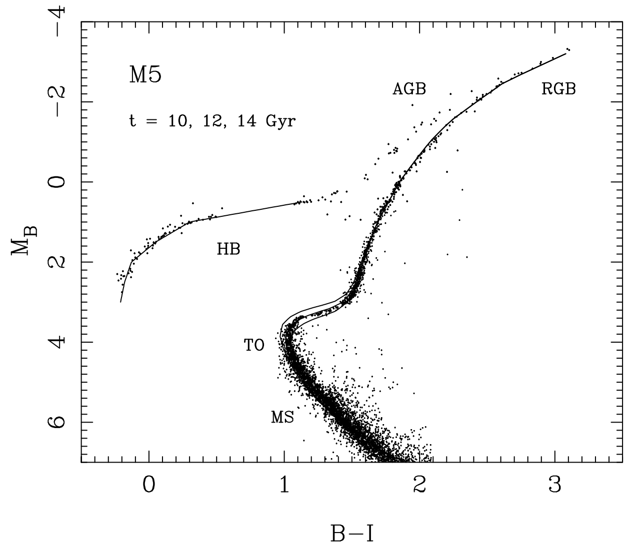

Figure 1: A colour–magnitude plot for stars in the globular cluster M5. This illustrates well the main features of stellar evolution: the main sequence (MS) and its turn-off point (TO), followed by the red giant branch (RGB), horizontal branch (HB) and asymptotic giant branch (AGB). The data are well-fitted by a theoretical isochrone of age 12 Gyr.

The giant branch: Once hydrogen is exhausted in the core, it continues to burn in a shell around the core. The helium-rich core begins to contract, and this core contraction is associated with an expansion in size of the outer envelope of the *turn-off star*, forming a subgiant. It would be nice if there existed a simple reason why giant stars increase in size so much, but one of the embarrassments of the subject is the lack of any such explanation. A common and deceptively simple argument is that it is all to do with the virial theorem, where the kinetic and potential energies must satisfy $2E_{\rm K} + E_{\rm V} = 0$ in equilibrium: since the contraction of the core causes $E_{\rm V}$ to become more negative, the balance can be restored if the outer parts expand to become less tightly bound.

In the initial phases of core contraction, the turn-off stars become somewhat bigger at constant luminosity (and hence have cooler surface temperatures). The helium-rich core continues to contract until it is supported by *electron degeneracy pressure*, which we discussed earlier. As the star burns more hydrogen, the mass and temperature of the helium core increase, and this higher temperature drives up the fusion rate in the hydrogen-burning shell. This leads to a vast increase in nuclear energy output, boosting greatly the luminosity of the star. The star's size also continues to grow (by a factor of 10–100), such the star ascends the *red giant branch* at roughly constant effective temperature.

Table 1 gives characteristic sizes and luminosities for red giant stars. Note that these depend largely on the spectral type, and not on the mass: stars of a wide range of masses follow similar tracks up the HR diagram, becoming redder and more luminous as they grow. This whole process is rather fast: the subgiant phase lasts roughly 10% of the main-sequence lifetime, and the red-giant phase is roughly 5% of the main-sequence lifetime.

Table 1: Radii and luminosities of red giants.

	G0	G5	K0	K5	M0	M5
$\log_{10}(R/R_\odot)$	0.8	1.0	1.2	1.4	1.6	1.9
$\log_{10}(L/L_\odot)$	1.5	1.7	1.9	2.3	2.6	3.0

The horizontal branch: The mass and temperature of the helium core continue to increase until, for stars more massive than $\approx 0.5M_\odot$, a temperature $T \gtrsim 10^8$K is reached and helium begins to undergo fusion. This begins concurrently across the whole of the degenerate core of the star, a process known as the *Helium Flash*. Helium burning is hindered by the absence of stable nuclei with masses 5 and 8, but it can fuse through a process known as the *triple-alpha process*, whereby three helium nuclei ultimately combine to produce a ^{12}C nucleus.

$$\begin{aligned} {}^4\mathrm{He} + {}^4\mathrm{He} &\rightleftharpoons {}^8\mathrm{Be} \\ {}^8\mathrm{Be} + {}^4\mathrm{He} &\rightleftharpoons {}^{12}\mathrm{C}^* \\ {}^{12}\mathrm{C}^* &\rightarrow {}^{12}\mathrm{C} \qquad + 7.367\ \mathrm{MeV} \end{aligned} \tag{1}$$

The ^{8}Be produced in the first step is unstable and decays back into two helium nuclei in about 3×10^{-16} seconds. However in the dense region of a stellar core there is a finite probability that another helium nucleus will collide with the ^{8}Be before it decays, producing an excited state of ^{12}C. Because this excited state has almost exactly the same energy as the ^{8}Be and ^{4}He combined, the cross-section for this reaction is relatively large, greatly increasing the reaction probability. The existence of this excited state was predicted by Fred Hoyle before its actual observation based on its necessity for carbon to be formed.

When helium burning commences, stars follow their usual pattern of defying intuition by reducing luminosity somewhat but becoming smaller and hotter. They end up more or less at the top end of the original main sequence, which is reassuring, since we argued that this was largely independent of the details of nuclear energy generation for stars where the energy originated in the core.

The asymptotic giant branch: History now repeats itself as helium in the core is exhausted. Shell burning returns, with both a helium-burning shell and an outer hydrogen-burning shell, and the star ascends the giant branch once more, now termed the *asymptotic giant branch*, although its location in the HR diagram is rather similar to the red giant branch.

Late stages of evolution: For stars above 8-10 solar masses, eventually the core becomes hot enough for carbon burning, and even heavier element burning, to take place. These are very short-lived phases, each one hotter and quicker than the previous one (the final fusion step lasts only about a day). Eventually no further nuclear energy can be generated, and the iron core of the star collapses. The collapsed core forms a *neutron star* or *black hole* (which we'll discuss later), while the outer envelope is blown away in a *supernova explosion*.

Lower mass stars do not get hot enough for carbon burning. When they come to the end of the AGB stage they contain high-density cores enhanced in heavy elements. The normal behaviour is for the remaining outer layers to be lost, in a greatly exaggerated analogue of the Solar wind, leaving behind a *white dwarf*, which simply cools via radiation without generating further nuclear energy. These stars lie below the main sequence, being of very low luminosity for their temperature.

2 Degeneracy pressure in stars

2.1 Density of states

When discussing the minimum mass required for an object to become a star, we noted that the uncertainty principle imposes a limit on the range of position that a particle with a given range of momentum may occupy ($\Delta x \Delta p_x \gtrsim \hbar$), and therefore to how tightly packed electrons of a given range of momenta can be made. This resulted in the electrons giving rise to a pressure, known as **electron degeneracy pressure**, which prevented further collapse of the gas cloud. As discussed above, this electron degeneracy pressure is also important in supporting the non-burning helium cores of stars in the latter stages of stellar evolution, and it is also what is responsible for supporting white dwarf stars. We will now calculate in detail the pressure that the electrons can provide, as a function of their density ρ_e.

To derive this, we start by considering the **density of states** for free electrons: how many free-electron states fit into a box of volume $V = L^3$. Briefly, the box should be imagined as filling all space, via periodic replication. This means that the wavevectors of the free-electron quantum states can only take certain values. If the electron wave function is $\psi \propto \exp(i\mathbf{k} \cdot \mathbf{x})$, where $\mathbf{k} = (k_x, k_y, k_z)$, then periodicity requires

$$k_x = n_x \frac{2\pi}{L} \quad \text{where} \quad n_x = 1, 2, \ldots \tag{2}$$

The allowed states therefore lie on a lattice with spacing $2\pi/L$, and so the density of states in k space is

$$dN = g \frac{L^3}{(2\pi)^3} d^3k; \qquad (d^3k \equiv dk_x dk_y dk_z), \tag{3}$$

where g is a degeneracy factor for spin *etc.* Using **de Broglie's relation**, the momentum of the electron relates to its wave vector via $\mathbf{p} = \hbar\mathbf{k}$, and so the density of states in k space can be converted to that in **momentum space** as

$$dN = g \frac{L^3}{(2\pi\hbar)^3} d^3p. \tag{4}$$

The number density of particles (i.e. per unit volume) with momentum states in the range of d^3p is then

$$\boxed{dn = g\,\frac{1}{(2\pi\hbar)^3}\,f(p)\,d^3p,} \tag{5}$$

which is independent of volume $V = L^3$. Here $f(p)$ is the **occupation number** of the mode – i.e. the number of particles in the box with that particular wavefunction.

For **bosons** (particles such as photons), the occupation number $f(p)$ is unrestricted. For example, for black-body radiation, which is a gas of photons, the occupation number is $f_{BB} = (\exp[\hbar\omega/kT] - 1)^{-1}$. For **fermions** (particles such as electrons, whose spin angular momentum is $\hbar/2$), things are completely different, and the **Pauli exclusion principle** says that

$$\boxed{f(p) \leq 1.} \tag{6}$$

This criterion immediately imposes a restriction on how dense an electron gas can be before it has to be treated in a manner very different from the classical one. Normally, the distribution of momenta of the particles would be treated as a **Maxwellian distribution**, in which each component of velocity has a Gaussian distribution with standard deviation σ:

$$d\text{Prob} = \frac{1}{(2\pi\sigma^2)^{3/2}}\exp(-v^2/2\sigma^2)\;d^3v, \tag{7}$$

where v is the particle velocity. The dispersion in velocities, σ, is related to the temperature by equipartition of energy: $m\sigma^2/2 = kT/2$. We can convert this to a number density of particles in a given range of momentum space, by multiplying by the total number density, n, and using $p = mv$:

$$dn = \frac{n}{(2\pi mkT)^{3/2}}\exp(-p^2/2mkT)\;d^3p. \tag{8}$$

Comparing with the general expression for the number density of particles in momentum space (Equation 5), we deduce that there is a **critical density**, at which the classical law would yield $f > 1$ (at $p = 0$ which is where Equation 8 peaks):

$$\boxed{n_{\text{crit}} = \frac{g(mkT)^{3/2}}{(2\pi)^{3/2}\hbar^3}.} \tag{9}$$

Note that a simpler way of expressing this is $\hbar/(mkT)^{1/2} \simeq n_{crit}^{-1/3}$: the momentum dispersion is $p^2 \sim mkT$, so this just says that the typical interparticle spacing cannot be smaller than the limit set by the uncertainty principle.

If we take a fixed density, the gas will be in the classical regime for high T, but quantum effects become important as we go towards zero T. This is illustrated in Figure 2.

Integrating Equation 5 over all momentum states gives the total number density of particles:

$$n = g\,\frac{1}{(2\pi\hbar)^3}\int f(p)\;d^3p, \tag{10}$$

In the limit of zero temperature, states are occupied only up to the **Fermi momentum**, p_F. Hence

$$n = g\,\frac{1}{(2\pi\hbar)^3}\int_0^{p_F} d^3p = g\,\frac{1}{(2\pi\hbar)^3}\,\frac{4\pi}{3}\,p_F^3. \tag{11}$$

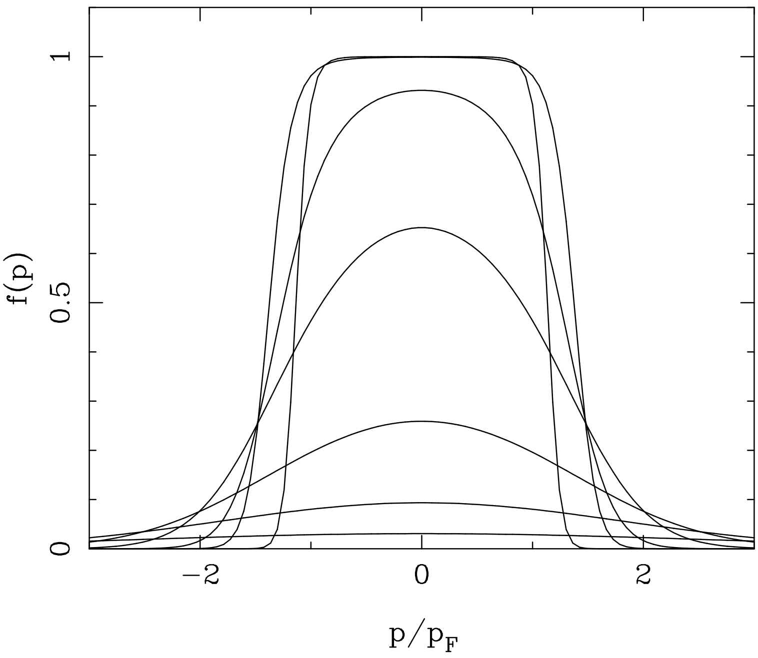

Figure 2: The occupation number for a gas of fermions as a function of their density relative to the critical density, ranging from $n/n_{\rm crit} = 0.03$ to $n/n_{\rm crit} = 30$. For the lowest densities (or highest temperatures), we have almost exactly the classical Maxwellian velocity distribution. For densities well above critical, the occupation number tends to a 'top-hat' distribution: unity for momenta less than the Fermi momentum, and zero otherwise.

The Fermi momentum is thus related to particle density by

$$\boxed{p_F = 2\pi\hbar \left(\frac{3}{4\pi g}\right)^{1/3} n^{1/3}} \tag{12}$$

As the density of the gas goes up, the Fermi momentum increases: the additional particles have to fill higher momentum states because the lower momentum states are fully occupied. For very high densities then the Fermi momentum can reach relativistic values (ie. some particles are forced into momentum states in which they have velocities approaching the speed of light). For this reason, in the sections below we consider the properties of both non-relativistic and relativistic gas.

2.2 Degeneracy pressure

The pressure of a degenerate electron gas can be worked out by using simple properties of each state, plus symmetry. Microscopically, pressure arises from flux of momentum. If the number density of electrons is n_e, then the flux of electrons in the x direction is just the number of electrons crossing unit area per unit time, or $n_e v_x$. The pressure is then approximately

$$P_e \simeq p_x \, n_e \, v_x, \tag{13}$$

where p_x is the momentum of the electrons.

The contribution to the total pressure in the x–direction from all of the electrons with momentum p_x is thus given by (cf. Equation 13)

$$dP_x = p_x \, v_x \, dn_{e,x} \tag{14}$$

where $dn_{e,x}$ is the number density of electrons with x–momentum in the range p_x to $p_x + dp_x$ (cf. Equation 5). We can then integrate over the density of states to deduce the pressure in the x direction (which by isotropy must be equal to the pressure in any direction, P).

$$P = P_x = g\,\frac{1}{(2\pi\hbar)^3}\int p_x v_x f(p) d^3p. \tag{15}$$

Note that (as can be regarded as obvious by isotropy, or could be derived using spherical polars in momentum space)

$$\int p_x v_x \, dp_x \, dp_y \, dp_z = \frac{1}{3}\int (p_x v_x + p_y v_y + p_z v_z) \, dp_x \, dp_y \, dp_z = \frac{1}{3}\int p.v \; 4\pi p^2 \, dp. \tag{16}$$

Therefore,

$$\boxed{P = \frac{g}{3}\,\frac{1}{(2\pi\hbar)^3}\int_0^\infty p.v \; f(p) 4\pi p^2 dp.} \tag{17}$$

Consider now the extreme non-relativistic and relativistic limits of the exact expressions for the pressure, in the zero-temperature case.

(1) non-relativistic case: Here $v = p/m_e$, so $p.v = p^2/m_e$. Thus,

$$P_e = \frac{4\pi g}{3(2\pi\hbar)^3}\int_0^{p_F} (p^2/m_e)\, p^2 \, dp = \frac{g}{30\pi^2\hbar^3 m_e} p_F^5. \tag{18}$$

From Equation 12, $p_F = 2\pi\hbar(3n_e/4\pi g)^{1/3}$. Thus, using $\rho_e = m_e n_e$, we have

$$p_F = \left(\frac{6\pi^2\hbar^3 \rho_e}{g m_e}\right)^{1/3} \tag{19}$$

and so

$$P_e = \frac{g}{30\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3}{g}\right)^{5/3} \rho_e^{5/3} m_e^{-8/3} \equiv K_1 \rho_e^{5/3} \tag{20}$$

where

$$K_1 = \frac{\pi^2\hbar^2}{5 m_e^{8/3}}\left(\frac{6}{g\pi}\right)^{2/3}. \tag{21}$$

(2) relativistic case: $v = c$, so $p.v = pc$, giving

$$P_e = \frac{4\pi g}{3(2\pi\hbar)^3}\int_0^{p_F} pc\, p^2 \, dp = \frac{gc}{24\pi^2\hbar^3} p_F^4 \tag{22}$$

$$= \frac{gc}{24\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3 \rho_e}{g m_e}\right)^{4/3} \equiv K_2 \rho_e^{4/3} \tag{23}$$

$$K_2 = \frac{\pi\hbar c}{4 m_e^{4/3}}\left(\frac{6}{g\pi}\right)^{1/3}. \tag{24}$$

Note that for both cases the electron degeneracy pressure P_e depends only on the electron density ρ_e. For partially degenerate gas, however, P_e will depend on both ρ_e and T. Note also that the degeneracy pressure depends on a negative power of m_e: the degeneracy pressure from the protons that are also present is therefore negligible, justifying us having considered only the electron component.

The functional form of these equations could have been derived in a more simple way, just using the uncertainty principle. Since $P_e \simeq p_x\, n_e\, v_x$, and in the non-relativistic case $v_x = p_x/m_e$, then $P_e \simeq n_e\, p_x^2/m_e = \rho_e\, p_x^2 m_e^{-2}$ (since $\rho_e = m_e n_e$). By the uncertainty principle, the minimum volume these particles may be squeezed into is $(\Delta x)^3 \simeq (\hbar/p_x)^3$, corresponding to a mass density $\rho_e \simeq m_e/(\Delta x)^3 = m_e(p_x/\hbar)^3$. Re-arranging this, $p_x \simeq \hbar(\rho_e/m_e)^{1/3}$, and so we obtain $P_e \simeq \rho_e p_x^2 m_e^{-2} \simeq \hbar^2 \rho_e^{5/3} m_e^{-8/3}$. Similarly in the relativistic case, $v_x = c$, giving $P_e \simeq p_x n_e c = p_x c \rho_e/m_e$; as before the uncertainty principle gives $p_x \simeq \hbar(\rho_e/m_e)^{1/3}$, so we have now $P_e \simeq \hbar c \rho_e^{4/3} m_e^{-4/3}$.

3 White dwarfs and the Chandrasekhar limit

Degeneracy pressure is important in massive stars as they evolve off the main sequence. We have appealed to the effect to hold up the cores once hydrogen burning ceases, and the same effect supports the white dwarfs that are formed at the very end of the evolutionary sequence. The maximum mass that a white dwarf can have (the **Chandrasekhar mass**) is about 1.4 $M_\odot$, and this arises roughly as follows.

The energy density of the gas can be calculated in exactly the same way as we obtained the number density and pressure: by integrating over momentum space, putting in the energy per mode, $\epsilon(p)$:

$$\boxed{U = g\,\frac{1}{(2\pi\hbar)^3}\int_0^\infty \epsilon(p)\; f(p)\; 4\pi\; p^2\; dp.} \tag{25}$$

Consider the energy density for zero-temperature, in the limit that the electrons are all highly relativistic (*i.e.* the Fermi momentum is $p_F \gg m_e c$, which will be true at high enough density; cf Equation 19). In the zero-temperature limit we set $f(p) = 1$ up to the Fermi momentum, and in the relativistic limit, $\epsilon(p) = pc$. Therefore the energy density is

$$U_e = g\,\frac{1}{(2\pi\hbar)^3}\,4\pi c p_F^4/4, \tag{26}$$

which can be expressed in terms of the number density (cf Equation 12) as

$$U_e = \frac{3}{4}\left(\frac{6\pi^2}{g}\right)^{1/3} \hbar c\, n_e^{4/3}. \tag{27}$$

Similarly, in the opposite limit of highly non-relativistic electrons, $\epsilon(p) = p^2/2m_e$ (where $p = m_e v$) and the energy density is

$$U_e = \frac{3\hbar^2}{10 m_e}\left(\frac{6\pi^2}{g}\right)^{2/3} n_e^{5/3}. \tag{28}$$

The total kinetic energy of the electrons is $E_K \propto U_e V$. Suppose the star is in the relativistic regime, then the total kinetic energy of the electrons is $E_K \propto U_e V \propto n_e^{4/3} V \propto M^{4/3}/r$. The gravitational energy is proportional to $-M^2/r$, so that the total energy can be written as

$$E_{\rm tot} = (AM^{4/3} - BM^2)/r \tag{29}$$

(where A and B are constants). There thus exists a critical mass where the two terms in the bracket are equal. If the mass is smaller, then the total energy is positive and will be reduced by making the star expand until the electrons reach the mildly relativistic regime and the star can exist as a stable white dwarf (see Section 3.1). If the mass exceeds the critical value, the binding energy increases without limit as the star shrinks: gravitational collapse has become unstoppable.

To find the exact limiting mass, we need to find the coefficients A and B above, where the argument has implicitly assumed the star to be of constant density. In this approximation, the kinetic and potential energies are

$$E_K = \left(\frac{243\pi}{128g}\right)^{1/3} \frac{\hbar c}{r} \left(\frac{M}{\mu m_p}\right)^{4/3}; \qquad E_V = -\frac{3GM^2}{5r}, \tag{30}$$

where the mass per electron is μm_p. This estimate of the critical mass (for $g = 2$) is

$$\boxed{M_{\rm crit} = \frac{3.7}{\mu^2} \left(\frac{2}{g}\right)^{1/2} \left(\frac{\hbar c}{G}\right)^{3/2} m_p^{-2} \simeq \frac{7}{\mu^2} M_\odot.} \tag{31}$$

For a star near the end of the evolutionary track, much of the initial fuel has been burned to elements heavier than Helium, so that $\mu \simeq 2$. In fact, more exact calculations show that the critical mass in this case is about 1.4 $M_\odot$.

3.1 Sizes and densities of white dwarfs

White dwarfs are extremely compact objects. To find their radius, we assume that the electrons are non-relativistic; their energy density is then $\propto n_e^{5/3}$, so the total kinetic energy is $E_K = CM^{5/3}R^{-2}$ (where C is a constant). As before, we write $E_V = -BM^2/R$, and the total energy is the sum of these terms. The equilibrium radius is where $dE/dr = 0$, which gives

$$R = (2C/B)M^{-1/3}, \tag{32}$$

The star therefore contracts as its mass is increased. To put numerical values into this, the electron number density is

$$n_e = (M/\mu m_p) \,/\, (4\pi R^3/3), \tag{33}$$

and we have the energy density in terms of the number density (Equation 28)

$$U_e = \frac{3\hbar^2}{10m_e} \left(\frac{6\pi^2}{g}\right)^{2/3} n_e^{5/3}, \tag{34}$$

which gives

$$C = \frac{3\hbar^2}{10m_e} \left(\frac{6\pi^2}{g}\right)^{2/3} (\mu m_p)^{-5/3} \, (4\pi/3)^{-2/3}. \tag{35}$$

Combined with our previous $B = 3G/5$, we get

$$R = \frac{3}{2} \left(\frac{6\pi^2}{g^2}\right)^{1/3} \frac{\hbar^2}{Gm_e(\mu m_p)^{5/3}} M^{-1/3}. \tag{36}$$

Expressed in terms of the Chandrasekhar mass, this is

$$\boxed{R = 3\sqrt{\pi/5}\frac{\hbar}{\mu m_p m_e}\left(\frac{2\hbar}{gcG}\right)^{1/2}\left(\frac{M}{M_{\rm crit}}\right)^{-1/3} \simeq 5975\left(\frac{M}{M_{\rm crit}}\right)^{-1/3} \text{ km},} \qquad (37)$$

where the last figure assumes $g = 2$ and $\mu = 2$. In other words, a Chandrasekhar-mass white dwarf is about the size of the Earth. This is an extremely high density, which we can evaluate as

$$\rho = \frac{M}{4\pi R^3/3} = \frac{125\mu m_p m_e^3 c^3}{192\pi^2\hbar^3}\left(\frac{M}{M_{\rm crit}}\right)^2 \simeq 10^{9.6}\left(\frac{M}{M_{\rm crit}}\right)^2 \text{ kg m}^{-3}. \qquad (38)$$

These expressions contain tedious numerical coefficients, which come from working out the constant-density model. Even if the expressions were simpler, there would be no point in memorising the exact coefficients, because the constant-density model is not exact. If we wanted to do better, we would need to solve for the internal structure of a white dwarf.

In summary, non-relativistic white dwarfs are stable over a range of masses, but as their mass increases, they shrink. By the uncertainty principle, this must force the electrons closer and closer to becoming relativistic, and the limit of this is a star in which the electrons are all relativistic, which has a unique mass. It appears that electron degeneracy pressure has no way of supporting more massive bodies.

3.2 Cooling and ages of white dwarfs

All these calculations have assumed that the white dwarf is of zero temperature. This is not true initially, as the white dwarf starts life as the core of an evolved star. Nevertheless, degeneracy pressure allows the white dwarf to be supported without fusion energy generation, so it rapidly cools, and the zero-temperature approximation becomes a good one.

As usual, the black-body approximation can be used to deduce an effective surface temperature in terms of luminosity:

$$L = 4\pi R^2\,\sigma\,T_{\rm eff}^4, \quad \text{where} \quad \sigma = 5.67\times 10^{-8}\ \text{W m}^{-2}\text{K}^{-4}; \qquad (39)$$

Using the mass–radius relation derived above (Equation 37), we can obtain the temperature corresponding to a given luminosity:

$$T_{\rm eff} = 10^{4.8}(L/L_\odot)^{1/4}(M/M_{\rm crit})^{1/6}\ \text{K}. \qquad (40)$$

Because they are so small, luminous white dwarfs must be extremely hot; conversely, white dwarfs with effective temperatures closer to that of the sun are very low-luminosity objects, and extremely hard to detect.

Such cool white dwarfs are interesting, because they are natural clocks: the coolest ones are the oldest. One might think of working out the age–temperature relation just by using the above temperature to calculate the thermal energy of the star, and equating the luminosity to rate of change of thermal energy. This goes wrong because the internal temperature tends to be much higher than the surface. The interior is at a single temperature owing to thermal conduction by the degenerate electrons, but the surface layer is non-degenerate and serves as an effective insulator. The lifetimes of white dwarfs are therefore longer than the naive sum would indicate – but still finite. There are claims of a cutoff in numbers at about $L = 10^{-4.5}L_\odot$, which would correspond to an age of assembly for the Milky Way disk of about 9 Gyr, but this is still an area of active research.

4 Neutron stars

For a white dwarf slightly more massive than the Chandrasekhar limit, it appears that gravitational collapse must follow, with a corresponding increase in density. As the density of free electrons and nuclei goes up, the reaction of **inverse beta decay** is favoured – i.e. an electron can combine with a proton in a nucleus to yield a neutron. Once the density reaches a critical level, neutron rich nuclei (like ^{118}Kr) start to release free neutrons, in the phenomenon of **neutron drip**. This happens at a density of $\rho \simeq \rho_{\rm drip} = 4 \times 10^{14}\,{\rm kg\,m^{-3}}$.

Being fermions, these neutrons can provide a degeneracy pressure just in the same way as the electrons. As more and more electrons combine with protons, their number density falls, and the number density of free neutrons rises. At a density of $\rho = 4 \times 10^{15}\,{\rm kg\,m^{-3}}$, half the pressure is provided by the neutrons. At $\rho \simeq 2.4 \times 10^{17}\,{\rm kg\,m^{-3}}$, the remaining nuclei touch, leaving essentially a giant ball of neutrons, with some electrons and protons mixed in. Effectively, the star has become a single nucleus, of colossal proportions. In short, a last-gasp strategy for evading total collapse of a white dwarf is to combine its electrons and protons into neutrons.

The same stability analysis could then be applied with neutrons as the fermions but now with $\mu \simeq 1$. This apparently gives a limiting mass 4 times larger for these **neutron stars** than for white dwarfs (cf. Equation 31). However, Newtonian analysis is no longer valid for neutron stars because the gravitational potential at the star's surface is relativistic. The best detailed calculations give a maximum neutron star masses and radii of

$$M_{\rm max} \simeq 2 - 3\ M_\odot \qquad R_{\rm max} \simeq 10 - 15\ {\rm km}. \tag{41}$$

These figures are still uncertain, however, owing to the exotic nature of neutron star material.

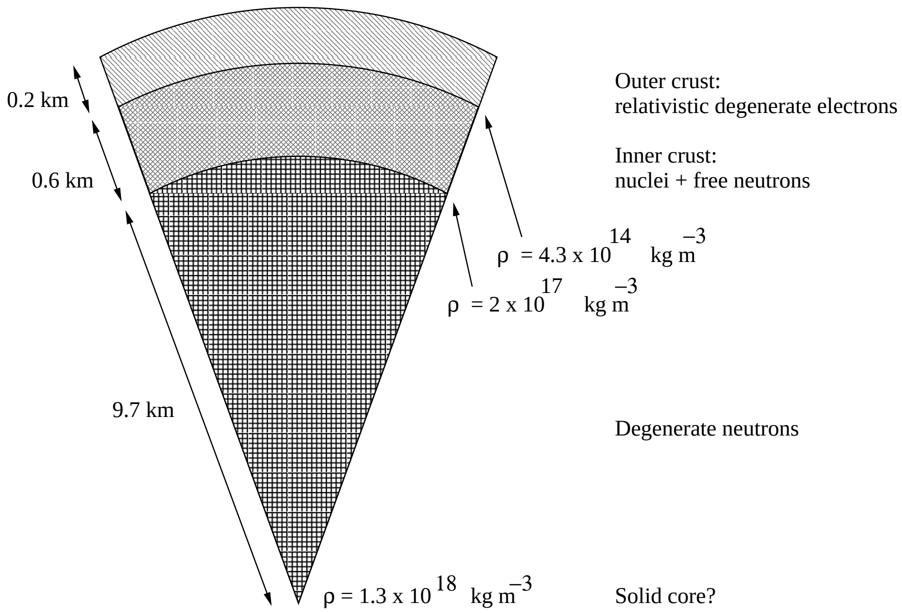

Figure 3: The density of a neutron star is highest at the centre and decreases outward. In the outer crust, the pressure is dominated by relativistic degenerate electrons. The majority of the volume, however, is dominated by a relativistic gas of degenerate neutrons.

5 Black holes

For a neutron star above the theoretical mass limit, there is no known equation of state of nuclear matter that can prevent collapse. What then will be the fate of a more massive star once it stops burning, and so loses its thermal pressure support to balance gravity? Usually the entire disruption of the star will result in a spectacular supernova. But the best models we have for supernovae predict that for progenitor (pre-supernova) stars that are sufficiently massive, a core more massive than $2\,M_\odot$ will remain. The only known answer is that such a star will undergo a catastrophic collapse that, according to General Relativity, is unstoppable, resulting in a singularity in space- time. We don't see the singularity, however. Instead we see a **black hole**.

This name reflects the fact that light cannot escape at all from a sufficiently strong gravitational field. By something of a coincidence, a Newtonian argument gives the right answer for this, in an argument due to Laplace. A particle emitted with velocity v at radius r has total energy

$$E_{\rm tot} = mv^2/2 - GMm/r, \tag{42}$$

which is negative if $v^2 < 2GM/r$, so that the particle cannot escape to infinity. Letting $v \to c$ implies that there is an **event horizon** of radius $2GM/c^2$ from within which light cannot escape. A proper calculation in general relativity shows that this conclusion is actually correct. Moreover, as physical systems approach this radius, gravitational time dilation slows the apparent ticking of clocks to zero, so that emission of photons effectively ceases: the central regions of a black hole cannot be detected directly. Instead, we rely on emission from hot material in orbit at radii of a few times the horizon radius. Here, the orbital speeds are a good fraction of c, and material being accreted can be heated to the point where it emits X-rays.