Astrophysics 3, Semester 1, 2011–12

Physics of Nebulae (2): The dynamic ISM

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1 Astrophysical Fluid Dynamics

So far, the ISM has been treated as static, but this is not realistic. At a minimum, we have seen that young stars can heat the ISM in their vicinity; such energy input would make the gas tend to expand. We therefore need to consider the processes that govern the time dependence of the density and velocity in a fluid.

1.1 The fluid approximation

Before getting on with this task, we should ask whether the normal concept of a fluid is valid in astrophysics. Ultimately, all fluids are a collection of independent particles. The smooth and continuous behaviour we associate with e.g. water arises because the mean free path of an individual particle is normally very short: it is the interactions between neighbouring particles that give the fluid the ability to show collective behaviour. Since the ISM is a plasma with a density that is very low by terrestrial standards, we need to worry about how large the mean free path really is. The mean free path is $\lambda = (n\sigma)^{-1}$, where n is the number density and σ the cross section. A rough estimate for the latter is to say that $\sigma = \pi r^2$ and two particles interact (scatter) if they become sufficiently close that the electrostatic potential energy exceeds the thermal energy:

$$\frac{e^2}{4\pi\epsilon_0 r} \gtrsim kT, \qquad \text{ie.} \qquad r \lesssim \frac{e^2}{4\pi\epsilon_0 kT}. \tag{1}$$

This then gives

$$\lambda = \frac{1}{n\sigma} \sim \frac{(4\pi\epsilon_0 kT)^2}{\pi e^4 n} \simeq (T/30\mathrm{K})^2\,(n/10^{10}\,\mathrm{m}^{-3})^{-1}\ \mathrm{m}. \tag{2}$$

This mean free path is usually small (compared to astrophysical distances), but not always. For material in the Solar wind near the Earth, λ exceeds the size of the Earth, so the plasma is effectively **collisionless**. Nevertheless, fluid-like behaviour is seen, and this is because magnetic fields can act as extra ‘glue’ to make the plasma act collectively.

1.2 Equations of motion

The key concepts governing the behaviour of a fluid are (1) the conservation of mass, and (2) the acceleration of fluid elements by pressure gradients. We will be content with studying these issues in one dimension – i.e. the velocity, density etc. will be taken to depend only on x, and be constant over the yz plane.

To deduce the equations of motion, consider a small box, of area A and thickness Δx. The amount of mass in this box is $M = \rho V = \rho\, A\, \Delta x$, where ρ is the mass density of the fluid. This mass can change through the difference between the rates at which mass flows in and out of the walls. The rate at which mass crosses unit area (the **flux density** of mass) is ρv, as is easily seen: in time Δt, the fluid moves a distance $v\Delta t$, and so a volume of fluid $A\, v(x)\, \Delta t$ crosses the wall of the box from the left, and hence the mass of fluid entering the box is $A\, v(x)\, \rho(x)\, \Delta t$. Similarly, a mass $A\, v(x+\Delta x)\, \rho(x+\Delta x)\, \Delta t$ leaves the right-hand wall of the box. The net change in mass in the box is given by the difference in these values. Since the volume of the box is fixed, this change in mass corresponds to a change in density

$$\Delta\rho = \frac{\Delta M}{V} = \frac{A\, v(x)\, \rho(x)\, \Delta t \;-\; A\, v(x+\Delta x)\, \rho(x+\Delta x)\, \Delta t}{A\Delta x}. \tag{3}$$

Noting that $v(x+\Delta x)\, \rho(x+\Delta x) \simeq v(x)\, \rho(x) + \Delta x\, d(v\rho)/dx$, and dividing through by Δt gives

$$\frac{\Delta\rho}{\Delta t} = \frac{v(x)\, \rho(x) - \Big(v(x)\rho(x) + \Delta x\, \frac{d(v\rho)}{dx}\Big)}{\Delta x}. \tag{4}$$

We can consider small intervals so that $\Delta\rho/\Delta t$ becomes the derivative of ρ with respect to time. This gives a very simple equation, called the **equation of continuity**

$$\boxed{\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}\,(\rho v).} \tag{5}$$

The only subtlety here is that the derivatives involved must be *partial*: d/dx at constant t and vice-versa.

The other equation we will need is an equation of motion, or force law. Here, it is easiest to consider a box that moves with the fluid (i.e. we work in a frame of reference in which the fluid is instantaneously at rest). Alternatively, we are considering a little parcel of fluid whose walls are defined by particles, so that no material crosses them. Nevertheless, momentum crosses the walls, since the pressure in the fluid on either side of the walls acts on them. The net force on the $+x$ direction is just the pressure acting on the left, minus that on the right, times the area A:

$$F = A\, P(x) - A\, P(x+\Delta x). \tag{6}$$

This force must equal the mass of the fluid element times its acceleration: $\rho\, A\, \Delta x\, \dot{v}$. Therefore, we get a simple equation of motion, known as **Euler's equation**:

$$\boxed{\dot{v} = -\frac{1}{\rho}\frac{\partial P}{\partial x}.} \tag{7}$$

The only subtlety here is the meaning of $\dot{v}$. This is a time derivative as seen by an observer who moves with the fluid, and it is a mixture of time and spatial derivatives as seen in the lab:

$$\dot{v} \equiv \frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} \tag{8}$$

(because in a time Δt, the fluid moves a distance $v\Delta t$). The idea here is that the changes experienced by an observer moving with the fluid are inevitably a mixture of temporal and spatial changes.

1.3 Sound waves

Now that we have the fundamental partial differential equations that govern fluid flow, we can perform a very important analysis and ask what happens if we **perturb** the fluid by changing its density etc. by small amounts:

$$\rho \to \rho + \delta\rho. \tag{9}$$

We further assume that the unperturbed state is as simple as possible: a fluid of uniform density ρ_0 at rest ($v_0 = 0$). The equation of continuity is

$$\begin{aligned}
\frac{\partial\,(\rho_0 + \delta\rho)}{\partial t} &= -\frac{\partial}{\partial x}\,(\rho_0 + \delta\rho)(v_0 + \delta v). \\
\Rightarrow \quad \frac{\partial\,\delta\rho}{\partial t} &= -\frac{\partial}{\partial x}\,(\rho_0 + \delta\rho)\delta v \\
\Rightarrow \quad \frac{\partial\,\delta\rho}{\partial t} &= -\rho_0\frac{\partial\,\delta v}{\partial x} - \frac{\partial\,\delta\rho\,\delta v}{\partial x}
\end{aligned} \tag{10}$$

We can make this even simpler by exploiting the fact that the perturbations are small: the last term contains the quadratic quantity $\delta\rho\,\delta v$, which must be negligible compared to $\rho_0\,\partial\,\delta v/\partial x$. The equation of continuity is then

$$\frac{\partial\delta\rho}{\partial t} = -\rho_0\frac{\partial\,\delta v}{\partial x}. \tag{11}$$

Similar reasoning makes Euler's equation simple: v_0 is zero, so the equation is

$$\frac{\partial\,\delta v}{\partial t} + \delta v\,\frac{\partial\,\delta v}{\partial x} = -\frac{1}{\rho_0 + \delta\rho}\,\frac{\partial\,\delta P}{\partial x}. \tag{12}$$

Neglecting all second-order terms, this becomes

$$-\rho_0\,\frac{\partial\,\delta v}{\partial t} = \frac{\partial\,\delta P}{\partial x} \tag{13}$$

We can eliminate δv from these equations (11 and 13) by taking the time derivative of the first and the space derivative of the second to yield $-\rho_0\,\partial^2\delta v/\partial t\partial x$ in both cases. This gives

$$\frac{\partial^2\delta\rho}{\partial t^2} = \frac{\partial^2\delta P}{\partial x^2}. \tag{14}$$

Finally, we need a relation between perturbations in pressure and in density. If we define a symbol

$$c_s^2 \equiv \frac{dP}{d\rho}, \tag{15}$$

then the equation becomes

$$\boxed{\frac{\partial^2\,\delta\rho}{\partial t^2} - c_s^2 \frac{\partial^2\,\delta\rho}{\partial x^2} = 0} \tag{16}$$

This is the one-dimensional **wave equation**, which has solutions that propagate at velocity c_s: $\delta\rho = f(x \pm c_s t)$. The quantity c_s is thus the **speed of sound** in the fluid.

We can think of two possible cases for evaluating c_s. Suppose the fluid was **isothermal** – i.e. at fixed temperature T, so that $P = (\rho/m)kT$, where m is a particle mass. In this case, $c_s^2 = kT/m$, which just says that the sound speed is of order the internal velocity dispersion of the particles. More plausibly, the equation of state will be **adiabatic**, since the fluid may not be able to radiate away extra energy gained when it is compressed. A better assumption is then $P \propto \rho^\gamma$, so that $c_s^2 = \gamma P/\rho$. For a simple gas, $\gamma = 5/3$, so that c_s is $\sqrt{5/3}$ greater than the isothermal case.

2 Shock waves

Despite the above analysis, not all waves in fluids have to travel at the speed of sound. Let's suppose we have a box of gas at rest. We then drive a piston into it, at first slowly: this will set up a pressure gradient which will accelerate the gas ahead of the piston (see Figure 1). The setting-up of the pressure gradient is done by sound waves: they communicate the presence of the piston to the fluid ahead. Therefore the fluid will be unaltered at $x > c_s\,t$. If the piston is pushed faster, it will start to catch up to the leading sound wave; as it does so, the gradients of pressure etc. in the fluid ahead become larger and larger. For example, $v(x)$ has to change from the piston velocity to zero over the distance from the piston position to $x = c_s t$, so dv/dx diverges as the piston approaches $x = c_s t$. What will happen if $v_{\rm piston} > c_s$? The sound waves will no longer be able to warn the fluid ahead, and yet somehow the fluid must know about the piston in order to respond. What does the fluid do? It responds by a discontinuous change in the fluid flow variables – a **shock**.

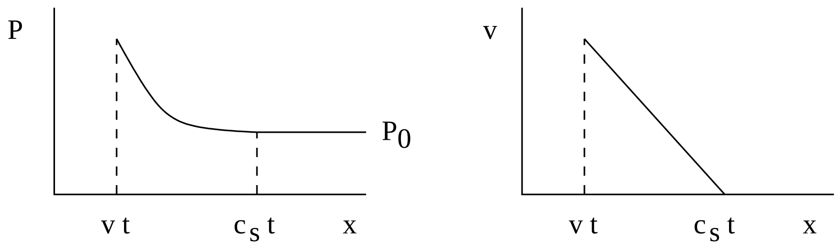

Figure 1: A piston is driven into a fluid at velocity v at $t = 0$, so that the piston position is $x = vt$. Pressure waves can move ahead of the piston at the speed of sound, c_s, so that there are gradients in pressure and velocity ahead of the piston, matching onto the undisturbed conditions (velocity zero; pressure P_0) for $x > c_s t$. As the piston's speed approaches $v = c_s$, these gradients become infinite, and a shock front is formed.

The laws of conservation must still apply, so we can relate the fluid variables on either side of the shock discontinuity, or shock front. Consider a shock front moving at speed $v_{\rm shock}$ into a stationary gas: it is simpler to look at this in the reference frame in which the shock front is stationary, so that upstream fluid arrives with velocity u_0 ($= v_{\rm shock}$), and streams away from the shock with velocity u_1 (strictly, u_0 and u_1 are negative, but it is convenient to reverse the x axis in the shock frame and they are treated as positive hereafter).

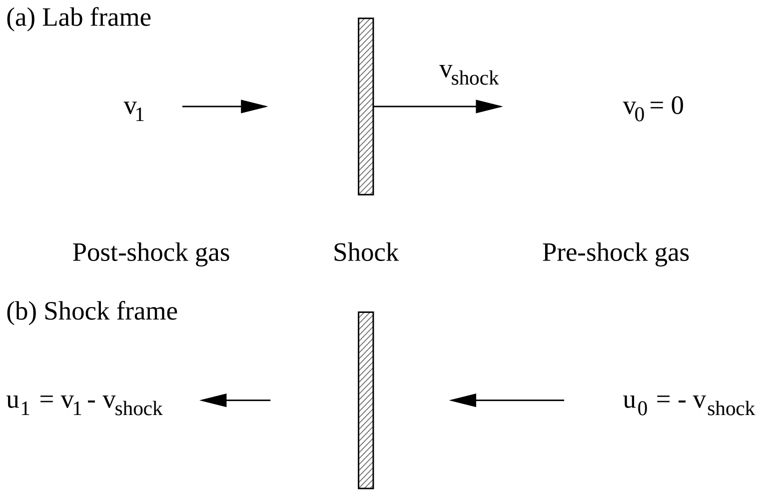

Figure 2: A plane shock front viewed in two reference frames: (a) the frame of the unshocked upstream fluid; (b) the frame in which the shock front is stationary.

If we consider a steady-state shock moving at constant velocity, the conservation laws in the shock frame are very simple: the amounts of mass, momentum and energy arriving per unit time per unit area of the shock from upstream, plus any extra amount generated at the shock, must equal what is transported away downstream.

Mass conservation The flux density of mass is density times velocity. No extra mass is generated at the shock, so that

$$\boxed{\rho_0 u_0 = \rho_1 u_1,} \tag{17}$$

in the shock frame, or $\rho_0 v_{\rm shock} = \rho_1 \left(v_{\rm shock} - v_1\right)$ in the lab frame.

Momentum conservation The flux density of momentum is momentum density (ρv) times velocity. Momentum is not conserved at the shock, however, since there will be a difference in pressure between upstream and downstream. This also generates momentum (because pressure equals rate of change of momentum per unit area). Thus, the momentum conservation law is

$$\boxed{\rho_0 u_0^2 + P_0 = \rho_1 u_1^2 + P_1} \tag{18}$$

in the shock frame, or $\rho_0 v_{\rm shock}^2 + P_0 = \rho_1 (v_{\rm shock} - v_1)^2 + P_1$ in the lab frame.

Energy conservation The flux density of energy is velocity times the density of energy, which is the density of kinetic energy ($\rho v^2/2$) plus the internal energy density, ϵ. Energy is not conserved at the shock, since the pressure does work on the fluid as it flows. The rate of working per unit area is pressure times velocity. The overall conservation equation in the shock frame is therefore

$$\boxed{u_0 \left(\frac{1}{2}\rho_0 u_0^2 + \epsilon_0\right) + P_0 u_0 = u_1 \left(\frac{1}{2}\rho_1 u_1^2 + \epsilon_1\right) + P_1 u_1} \tag{19}$$

We can simplify this in two ways. First, note that $\epsilon = 3nkT/2 = 3P/2$ for a gas of simple particles, so that $\epsilon + P = 5P/2$. Second, we divide by the equation of conservation of mass ($\rho_0 u_0 = \rho_1 u_1$), which gives

$$\boxed{\frac{1}{2}u_0^2 + \frac{5}{2}\frac{P_0}{\rho_0} = \frac{1}{2}u_1^2 + \frac{5}{2}\frac{P_1}{\rho_1}.} \tag{20}$$

In the lab frame, this becomes

$$\frac{1}{2}v_{\rm shock}^2 + \frac{5}{2}\frac{P_0}{\rho_0} = \frac{1}{2}\left(v_{\rm shock} - v_1\right)^2 + \frac{5}{2}\frac{P_1}{\rho_1} \tag{21}$$

We thus have three relations for the three unknowns, u_1, ρ_1, P_1 in terms of u_0, ρ_0, P_0. This can be straightforwardly solved, but the expressions are cumbersome. For our purposes we will make the simplifying assumption that the shock is **strong**:

$$u_0^2 \gg \frac{P_0}{\rho_0} \qquad \Rightarrow \qquad u_0 \gg c_s(\text{upstream}), \tag{22}$$

so that the shock is **hypersonic**. The last conclusion follows from our previous discussion about sound speeds: $c_s^2 = \gamma P/\rho$ for adiabatic waves, or the same without the γ factor for isothermal waves. With this assumption, the term P_0 can be dropped from Equations 18 and 20. Then, re-arranging these equations, together with Equation 17, to eliminate P_1, ρ_0 and ρ_1 gives a quadratic equation involving u_1 and u_0:

$$u_0^2 - 5u_0u_1 + 4u_1^2 = 0 \tag{23}$$

One solution to this quadratic is $u_1 = u_0$, which is the condition in the absence of any shock. The other solution gives rise to the following key relations for a strong shock:

$$\boxed{\frac{\rho_1}{\rho_0} = 4; \quad P_1 = \frac{3}{4}\rho_0 u_0^2; \quad u_1 = \frac{1}{4}u_0.} \tag{24}$$

In terms of v_1 and $v_{\rm shock}$, noting $u_0 = -v_{\rm shock}$ and $u_1 = v_1 - v_{\rm shock}$, we obtain

$$P_1 = \frac{3}{4}\rho_0 v_{\rm shock}^2; \quad v_1 = \frac{3}{4}v_{\rm shock}, \tag{25}$$

plus an alternative form of the condition for a shock to be strong:

$$\frac{P_1}{P_0} = \frac{3}{4}\frac{v_{\rm shock}^2}{P_0/\rho_0} \gg 1. \tag{26}$$

In summary, there are three equivalent criteria that determine whether or not a shock is strong: (1) it is hypersonic; (2) it has a large pressure jump ($P_1/P_0 \gg 1$); (3) the **ram pressure** ($\rho_0 u_0^2$) greatly exceeds the upstream thermal pressure, P_0.

These results present a paradox: the shock compresses the gas, and our first thought might be that this process would be adiabatic, implying

$$\frac{P_1}{P_0} = \left(\frac{\rho_1}{\rho_0}\right)^{5/3} = 4^{5/3}, \tag{27}$$

whereas we have just shown that the pressure ratio across a hypersonic shock is $\gg 1$. The solution is that the compression is **irreversible** and hence not adiabatic: entropy is generated through viscous dissipation at the shock (nevertheless, confusingly, these are called **adiabatic shocks**, in order to distinguish them from the **isothermal shocks** studied below). The presence of viscosity is neglected in the above equations, but this is not correct where the fluid gradients are very high. The consequence of this is that the shock front is not really an abrupt discontinuity, but a continuous transition with a width of order the collisional mean free path.

2.1 Isothermal shocks

Often the gas radiates so strongly it maintains nearly a constant temperature on passing through the shock (see Figure 3). For such shocks, the equations of mass continuity ($\rho_1 u_1 = \rho_0 u_0$) and conservation of momentum ($P_1 = \rho_0 u_0^2 - \rho_1 u_1^2$; again assuming the strong-shock condition $\rho_0 u_0^2 \gg P_0$) still hold, but because of the radiative losses, the equation derived above for energy conservation is no longer applicable. Instead, for isothermal shocks, the constant temperature condition means that by the ideal gas law,

$$\frac{P_1}{\rho_1} = \frac{P_0}{\rho_0} = \frac{kT}{\mu m_H} = c_0^2 \tag{28}$$

where c_0 is the initial sound speed.

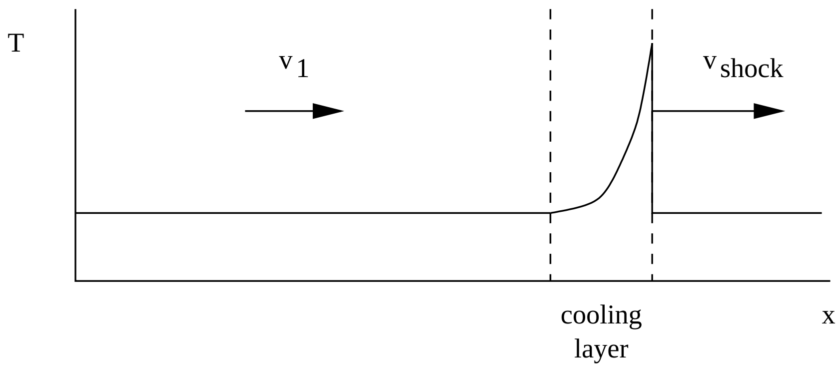

Figure 3: An isothermal shock is one where the post-shock temperature calculated according to the usual adiabatic shock formulae is so high that radiative cooling is very rapid. The temperature can then return to virtually the pre-shock temperature within a short distance. This 'cooling layer' is treated as being negligibly thin, and constituting an effective shock front for an isothermal shock.

To solve the equations for isothermal shocks, divide the momentum equation by the continuity equation, which gives

$$\frac{P_1}{\rho_1 u_1} + u_1 = u_0. \tag{29}$$

Dividing by u_1, we get

$$\frac{u_0}{u_1} = 1 + \frac{P_1/\rho_1}{u_1^2} = 1 + \frac{c_0^2}{u_0^2}\left(\frac{u_0}{u_1}\right)^2; \tag{30}$$

This is a quadratic equation for u_0/u_1. In the limit of $\frac{u_0}{c_0} \to \infty$, the solution of the quadratic is

$$\frac{u_0}{u_1} \quad \to \quad \left(\frac{u_0}{c_0}\right)^2. \tag{31}$$

Using continuity again, the density ratio is

$$\boxed{\frac{\rho_1}{\rho_0} \simeq \left(\frac{v_{\rm shock}}{c_0}\right)^2 \gg 1.} \tag{32}$$

Isothermal shocks can thus be extremely compressive, in contrast to adiabatic shocks. We now find $P_1 \simeq \rho_0 u_0^2$, instead of $\frac{3}{4}\rho_0 u_0^2$ previously, and can also write

$$\frac{P_1}{P_0} = \frac{\rho_1}{\rho_0} = \left(\frac{v_{\rm shock}}{c_0}\right)^2 \gg 1, \tag{33}$$

as the criterion for a strong shock.

Sound waves and shocks are important in the interstellar medium in a number of situations. Massive O & B stars produce **winds** with velocities as high as $v_* = 2000\mathrm{km\,s^{-1}}$, because their outer layers are unstable to **radiation pressure**, as we discussed earlier in the course: the momentum transferred to material that intercepts photons can be high enough to overcome gravity. Such stars can lose mass at rates as high as $\dot{M}_* \simeq 10^{-6} M_\odot\,\mathrm{yr}^{-1}$, giving a typical **mechanical luminosity** of $\dot{E}_* \equiv \frac{1}{2}\dot{M}_* v_*^2 \simeq 10^{29}$ W. The high velocity gas drives a shockwave into the ISM, giving rise to a hot bubble, even hotter than an HII region. In general, such winds are confined to the ionised interior of the HII region also produced by the stars.

A more dramatic input of energy into the ISM comes from supernovae. These are colossal stellar explosions, that propel material into the interstellar medium at speeds of up to $0.01c$.

3 Supernova Remnants

3.1 Types of supernovae

Supernovae (SNe for short) occur once every 30 years or so in a typical galaxy, and can briefly outshine all the stars in the galaxy that hosts them. Supernovae come in two-and-a-bit varieties, SNe Ia, Ib and II, distinguished according to whether or not they display absorption and emission lines of hydrogen. The SNe II do show hydrogen emission lines; they are associated with massive stars at the endpoint of their evolution, and are rather heterogeneous in their behaviour. The SNe I class do not show hydrogen emission lines. SNe Ia have very homogeneous properties, showing a characteristic rise in their light-curves to a maximum, followed by a symmetric fall over roughly 30 days, after which time the light decay becomes less rapid. The favoured model for these supernovae is that the explosion results from a white dwarf that has accreted material from a companion star. The lack of hydrogen in a white dwarf would account for the absence of hydrogen lines in the spectra of the SNe. Since white dwarfs are all close to the Chandrasekhar mass, the explosions tend to be similar, and this "standard candle" nature makes them especially important in cosmology as it allows us to infer their relative distances and thus establish the extragalactic distance scale (see next Semester for more details). Type Ib SNe are a complication to the scheme: although they also lack hydrogen lines, they do not have the characteristic light curve.

3.2 Initial expansion: blastwave

A supernova ejects about half its mass in the initial explosion, propelling its outer shells into the ISM with typical speeds of $0.01c$. The total kinetic energy of the ejecta is $\frac{1}{2}M_{\rm ej}v_s^2 \simeq 10^{43} - 10^{44}$ J for $M_{\rm ej} \simeq 4M_\odot$, where v_s is the initial velocity of the supernova shock.

We can use the solutions to the shock equations to determine the behaviour of the supernova during its initial expansion phase. Let ρ_0 be the density of the gas surrounding the supernova. For an adiabatic shock, the post shock pressure is (cf. Equation 25):

$$P_1 = \frac{3}{4}\rho_0\, v_s^2, \tag{34}$$

so that the thermal energy per unit mass is

$$\epsilon_T = \frac{3}{2}\frac{P_1}{\rho_1} = \frac{9}{8}\frac{\rho_0}{\rho_1}v_s^2 = \frac{9}{32}v_s^2 \tag{35}$$

(since $\rho_1/\rho_0 = 4$ and thermal energy is $\frac{3}{2}kT$ per particle). The kinetic energy per unit mass in the post-shock gas is

$$\epsilon_K = \frac{1}{2}\left(\frac{3}{4}v_s\right)^2 = \frac{9}{32}v_s^2 \tag{36}$$

(since $v_1 = 3v_s/4$). The amount of mass swept up by the shock is

$$M_s = \frac{4}{3}\pi\rho_0 R^3, \tag{37}$$

where R is the radius to which the shock has expanded. Hence, the total energy is

$$E_{\mathrm{tot}} = \frac{4}{3}\pi R^3\rho_0\,(\epsilon_T + \epsilon_K) = \frac{3}{4}\pi\rho_0 R^3 v_s^2. \tag{38}$$

Now, $v_s = \dot{R}$, and $E_{\mathrm{tot}} = E_{SN} =$ blast energy, so

$$E_{SN} = E_{\mathrm{tot}} = \frac{3\pi}{4}\,\rho_0 R^3\dot{R}^2. \tag{39}$$

We have a differential equation for R in terms of E_{SN}, ρ_0, and t.

$$R^{3/2}\frac{dR}{dt} = \left(\frac{4}{3\pi}\frac{E_{SN}}{\rho_0}\right)^{1/2} \tag{40}$$

Integrating both sides gives

$$\frac{2}{5}R^{5/2} = \left(\frac{4}{3\pi}\frac{E_{SN}}{\rho_0}\right)^{1/2} t \tag{41}$$

or

$$\boxed{R = \left(\frac{25}{3\pi}\right)^{1/5}\left(\frac{E_{SN}}{\rho_0}\right)^{1/5} t^{2/5}.} \tag{42}$$

Knowing $R(t)$, we differentiate to get the shock speed:

$$v_s = \dot{R} = \frac{2}{5}\left(\frac{25}{3\pi}\right)^{1/5}\left(\frac{E_{SN}}{\rho_0}\right)^{1/5} t^{-3/5}. \tag{43}$$

Taking $E_{SN} = 10^{44}$ J and $n_0 = 10^6\,\mathrm{m}^{-3}$ (giving $\rho_0 \simeq 0.025 M_\odot\,\mathrm{pc}^{-3}$), we get

$$\begin{aligned} R &\simeq 0.35\, t_{\mathrm{yr}}^{2/5}\ \mathrm{pc} \\ \dot{R} &\simeq 140,000\, t_{\mathrm{yr}}^{-3/5}\ \mathrm{km\,s^{-1}} \\ M_{\mathrm{shell}} = \frac{4}{3}\pi R^3 \rho_0 &\simeq 0.004\, t_{\mathrm{yr}}^{6/5}\ M_\odot \end{aligned} \tag{44}$$

Note that this solution must break down at early times, as it predicts a velocity that diverges as $t \to 0$. In fact, the similarity solution only starts to apply after a few hundred years, where $v_s \simeq 0.01c$ is predicted (the initial ejection velocity). At this time, the predicted shell mass is of order the mass of the original ejecta: another way of looking at this is that the similarity solution sets in once the blastwave has swept up at least as much mass as was originally ejected.

3.3 Radiative phase

Initially the post-shock gas temperature is

$$kT_1 = \frac{3}{32} m_H v_s^2 \tag{45}$$

(allowing for $\mu = 1/2$ since the gas is made up of e^- & H^+), so $T_1 \simeq 10^8$ K for $v_s = 3000\mathrm{km\,s^{-1}}$. The temperature decreases with time like v_s^2. Combining Equations 45 and 43,

$$T_1 = \frac{3}{32}\frac{m_H}{k}\frac{4}{25}\left(\frac{25}{3\pi}\right)^{2/5}\left(\frac{E_{SN}}{\rho_0}\right)^{2/5} t^{-6/5} \simeq 5\times 10^7 t_{1000\,\mathrm{yrs}}^{-6/5}\ \mathrm{K}. \tag{46}$$

This assumes an adiabatic shock – i.e. that radiative cooling behind the shock can be neglected. To see whether this is true, we have to consider again the interstellar cooling curve, reproduced in Figure 4. Remember here that we expressed the volume emissivity, ϵ (power radiated from unit volume) as $\epsilon = \Lambda(T)\, n_H^2$, where the density-squared factor is a natural factor in any collisional processes and the residual dependence is on temperature alone. As discussed earlier, the cooling curve has a sharp peak at $T \simeq 3\times 10^5$ K, due to cooling by O^+ ions. Once the post-shock material reaches this temperature, it will therefore very rapidly radiate away all its internal energy, to reach a temperature of order 10^4 K – similar to much of the undisturbed ISM, for the same reason. From Equation 46 this occurs at a time of $t \simeq 70,000$ yrs. At this time the supernova has a size $R \simeq 30$ pc and the shock velocity is $v_s \simeq 170\mathrm{km\,s^{-1}}$.

The **cooling time**, i.e. the time taken for gas to radiate away all of its energy, is given by

$$t_{\mathrm{cool}} \approx \frac{(2n_H)\frac{3}{2}kT}{n_H^2 \Lambda(T)}. \tag{47}$$

At the peak of the cooling curve, where $\Lambda(T) \simeq 10^{-34}\,\mathrm{W\,m^3}$ then for $n = n_p = n_e = 10^6\,\mathrm{m}^{-3}$,

$$t_{\mathrm{cool}} \simeq 10^{11}\ \mathrm{s} \simeq 4000\ \mathrm{yr} \tag{48}$$

In this time, the shock moves a distance

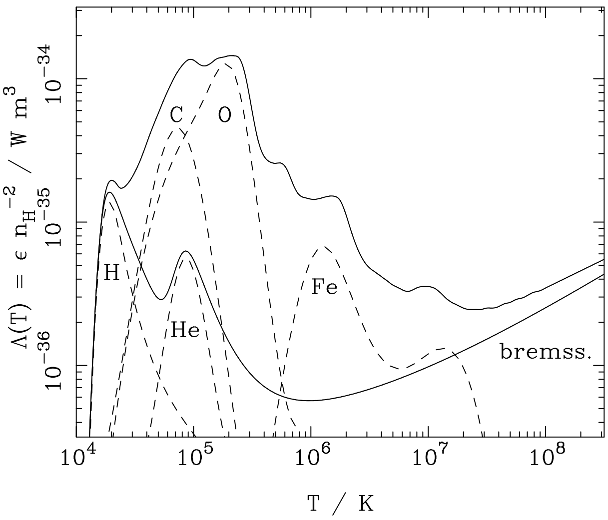

Figure 4: The cooling curve, defined such that the volume emissivity of the plasma, ϵ is $\epsilon \equiv \Lambda(T) n_H^2$. The two curves show results for hydrogen+helium only, and including Solar metals. The cooling curve shows a sharp peak near 10^5K.

$$\text{cooling length} \equiv \ell_{\text{cool}} = v_s\, t_{\text{cool}} \simeq 1\,\text{pc} \tag{49}$$

and picks up an additional amount of material

$$M_{\text{cool}} = 4\pi R^2\, \ell_{\text{cool}}\, \rho_0 \simeq 300 M_\odot \gg M_{\text{ej}}. \tag{50}$$

Much of the mass of the **supernova remnant (SNR)** is now in this thin shell, which radiates away most of its energy (and pressure) in optical light.

3.4 Momentum-conserving ('snowplough') phase

Although the blastwave pressure has been radiated away, and the supernova no longer has a shock-front, it continues to expand by momentum conservation.

The total mass in the SNR at this time is

$$M_{SNR} = \frac{4\pi}{3} R^3 \rho_0 \tag{51}$$

which for $R = 30$ pc and $n_0 = 10^6\,\text{m}^{-3}$ is about $3000 M_\odot$. The momentum is given by $\mathcal{M}_0 = M_{SNR} v_s$, and is conserved, ie.

$$\mathcal{M}_0 = \frac{4\pi}{3} R^3 \rho_0 \dot{R} = \text{constant}. \tag{52}$$

If we wait long enough that the size and mass of the SNR are much greater than the size and mass at the end of the radiative cooling phase, then integrating this gives

$$\boxed{R \simeq \left(\frac{3}{\pi}\right)^{1/4} \left(\frac{\mathcal{M}_0}{\rho_0}\right)^{1/4} t^{1/4}} \tag{53}$$

and

$$\dot{R} \simeq \frac{1}{4}\left(\frac{3}{\pi}\right)^{1/4}\left(\frac{\mathcal{M}_0}{\rho_0}\right)^{1/4} t^{-3/4} \simeq 4000\, t_{1000\ \mathrm{yr}}^{-3/4}\, \mathrm{km\,s^{-1}} \tag{54}$$

Eventually, $\dot{R}$ reaches the ISM sound speed, and the SNR dissipates. For $c_s^2 = (5/3)kT/\bar{m}$, and taking $T = 10^4$ K and $n_0 = 10^6\,\mathrm{m^{-3}}$, we get $c_s \simeq 17\mathrm{km\,s^{-1}}$, so the SNR dies after about 10^6 years, with a radius of order 100 pc.

3.5 The filling factor of SNe

What fraction of the disk of the Milky Way is in a SNR? We have calculated that a typical SNR radius is 100 pc and will last about 10^6 years before dissipating. We estimate about 1 SN every 30 yrs, so at any one time there will be on average $N_{SNR} = 10^6/30 = 3.3 \times 10^4$ SNR present in the galaxy. The volume occupied by these SNR is then

$$V_{SNR} = N_{SNR} \times \frac{4\pi}{3} R_{SNR}^3 = 140\ \mathrm{kpc}^3. \tag{55}$$

The radius of the Galaxy is $\simeq 15$ kpc, and its thickness is about 200 pc = 0.2 kpc, so its volume is $\simeq \pi(15\,\mathrm{kpc})^2 \times 0.2\,\mathrm{kpc} = 140\,\mathrm{kpc}^3$.

The mean number of SNR at any given point is thus:

$$\frac{V_{SNR}}{V_{\mathrm{Galaxy}}} \simeq 1. \tag{56}$$

Since SNR can overlap, the volume of the Galaxy in SNR is actually smaller: Poisson statistics indicate that the probability of there being no SNR at a given point is $\exp(-v_{SNR}/v_{\mathrm{Galaxy}})$, and so the fraction of the Galaxy filled by one or more SNR is

$$f_{SNR} = 1 - \exp(-v_{SNR}/v_{\mathrm{Galaxy}}) \simeq 0.6 \tag{57}$$

So most of the Galaxy, including probably us, is in a SNR.

This sum only counts the *active* SNR (i.e. those that exploded in the last 10^6 years). Since the galaxy is of order 10^{10} years old, we see that our neighbourhood has been affected by SNe at least 1000 times. One of the most important functions of supernovae in the ecology of the galaxy is to distribute the heavy elements that are made in the precursor star, thus **enriching** the ISM. This is the reason that the Sun contains metals even though it is young: it formed from gas that had been enriched by supernova explosions. The same also applies to the planets, which only form because the pre-Solar nebula is already rich in metals.