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Physics of Stars (2): Main Sequence Stars

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1 Solving the equations of stellar structure

1.1 Simple approximations and polytropic models

Before even thinking about how to solve the equations of stellar structure, it is important to emphasise that the key element in understanding the properties of stars is that the energy emerges via diffusion, and that this allows the systematic properties of the main sequence to be understood very simply.

Consider for simplicity a star of uniform density. Introducing a density profile only changes things by some dimensionless constants. In **virial equilibrium** the potential and kinetic energies are of the same order: $GM^2/R \sim nkTR^3$, so that $T \propto M/R$. Now, if photons suffer Thomson scattering inside the star with mean free path l $(= (n\sigma_T)^{-1} \propto R^3/M)$, the typical time for escape is that taken to diffuse a distance R, $t \sim R^2/cl$ (because the distance diffused in a random walk of N scatterings is $l\sqrt{N}$). The luminosity corresponds to radiating away the energy content $E \propto R^3T^4$ over this time, so $L \propto RT^4 l \propto R^4T^4/M$. Using the virial temperature above, this is just $L \propto M^3$. Putting in the dimensional constants gives

$$\boxed{L \simeq \frac{G^4 m_p^5}{\hbar^3 c^2 \sigma_T}\, M^3.} \tag{1}$$

In deducing this expression, we have made no assumptions at all about the internal nuclear reactions, which is quite remarkable.

Another way of writing L is to say that the surface radiates like a black body, $L \propto R^2T_{\rm eff}^4$. If we assume that the virial scaling also applies to $T_{\rm eff}$, $T_{\rm eff} \propto M/R$, we additionally deduce $T_{\rm eff} \propto M^{1/2}$, $R \propto M^{1/2}$. These relations in fact apply reasonably well to stars several times more massive than the sun. They fail for the sun (i) because some energy is transported via convection; (ii) because cooler stars have their opacity dominated by ionic processes: free–free and bound–free scattering, where *Kramer's opacity* applies.

An alternative simple approach uses the so–called **Polytrope Models**. The equation of hydrostatic equilibrium can be combined with that of the conservation of mass to eliminate $M(r)$ and derive the following:

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho \tag{2}$$

This equation involves two unknown functions, $P(r)$ and $\rho(r)$, but if a relationship between P and ρ is assumed then this can be reduced to a second–order differential equation in one variable. Polytropic models assume that pressure and density are related as:

$$P = K\rho^{1+1/n} \tag{3}$$

where K is a constant and n is the *polytropic index*. Substituting this into Equation 2,

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{d}{dr}\left(K\rho^{1+1/n}\right)\right] = -4\pi G\rho. \tag{4}$$

This differential equation can be solved uniquely if two boundary conditions are known. One such condition is that $\rho = \rho_C$ at $r = 0$. The second condition arises from the equation of hydrostatic equilibrium, which gives that $dP/dr = 0$ at $r = 0$: because of the polytropic relation between P and ρ, it follows that $d\rho/dr = 0$ also holds at $r = 0$. Using these boundary conditions, Equation 4 can be solved to determine $\rho(r)$ at all radii. It is important to note, however, that this can only be solved numerically, and also that the polytropic model is a significant simplification.

1.2 The dimensionless approach

Contrast the simple approach with a more mathematical approach using all four equations of stellar structure. The key idea here is one of **dimensionless variables**: given that the differential equations cannot be solved exactly, and must be solved numerically by computer, the quantities involved must be pure numbers, whereas M, R etc. have units. The solution is to make dimensionless versions of these variables by dividing by the natural unit supplied by the star itself. In some cases, this is the value at the surface, so we define

$$\tilde{r} \equiv r/R; \quad \tilde{M}(\tilde{r}) \equiv M(r)/M; \quad \tilde{L}(\tilde{r}) \equiv L(r)/L; \tag{5}$$

but the natural values of T, P and ρ are the central values:

$$\tilde{T} \equiv T/T_C; \quad \tilde{P} \equiv P/P_C; \quad \tilde{\rho} \equiv \rho/\rho_C; \tag{6}$$

Rewriting the equations of stellar structure in these units creates dimensionless combinations of the reference values. For example, consider the equations of continuity and hydrostatic equilibrium:

$$\frac{dM(r)}{dr} = 4\pi r^2\rho(r) \qquad \Rightarrow \qquad \frac{d\tilde{M}(\tilde{r})}{d\tilde{r}} = \left(\frac{R^3\rho_C}{M}\right) 4\pi\tilde{r}^2\tilde{\rho}(\tilde{r}) \tag{7}$$

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM(r)}{r^2} \qquad \Rightarrow \qquad \frac{1}{\tilde{\rho}}\frac{d\tilde{P}}{d\tilde{r}} = -\left(\frac{GM\rho_C}{RP_C}\right)\frac{\tilde{M}(\tilde{r})}{\tilde{r}^2} \tag{8}$$

The terms in brackets must be some dimensionless constants, and this is very important: *we only need to solve the equations once*. Suppose we integrate the differential equations starting from the centre, at $\tilde{r} = 0$. Here, $\tilde{M} = \tilde{L} = 0$ and $\tilde{T} = \tilde{P} = \tilde{\rho} = 1$. At the edge, we want to have reached $\tilde{M} = \tilde{L} = 1$ and $\tilde{T} = \tilde{P} = \tilde{\rho} = 0$. This will only happen if we choose the right values for dimensionless combinations like $(R^3\rho_C/M)$.

Solving the equations of stellar structure to deduce the correct dimensionless numbers can be complicated, and we can't go into much detail. A simple start is to choose trial forms for the desired functions that satisfy the boundary conditions. For example, in the equation of continuity we have to satisfy

$$\frac{d\tilde{M}(\tilde{r})}{d\tilde{r}} = \left(\frac{R^3\rho_C}{M}\right) 4\pi\tilde{r}^2\tilde{\rho}(\tilde{r}) \tag{9}$$

which requires $d\tilde{M}(\tilde{r})/d\tilde{r}$ to vanish at both $\tilde{r} = 0$ and $\tilde{r} = 1$. A simple form that achieves this is $d\tilde{M}(\tilde{r})/d\tilde{r} = A\tilde{r}^2(1 - \tilde{r})$, and $\tilde{M}(1) = 1$ gives $A = 12$. Solving the equation then gives $\tilde{\rho} = (1 - \tilde{r})$ and $(R^3\rho_C/M) = 12/4\pi$. Proceeding similarly with the other equations gives initial guesses for the other dimensionless parameters. We can then try to integrate the equations numerically. At the first pass, the exact boundary conditions will not be satisfied, but we will have a more accurate approximation for $\tilde{\rho}(\tilde{r})$ etc. The parameter guesses can be improved, and then the whole process iterated.

The end result of such a calculation comes quite close to more sophisticated calculations of the interior conditions in the Sun, which are reproduced below in Table 4.

Table 4 The Solar interior

$r/R_\odot$	$M(r)/M_\odot$	$\rho/\bar{\rho}$	$T/T_{\rm eff}$	$L(r)/L_\odot$
0	0	115	2,740	0
0.02	0.001	104	2,700	0.01
0.09	0.057	68	2,360	0.36
0.22	0.399	20.4	1,520	0.97
0.32	0.656	6.9	1,110	1.00
0.52	0.908	0.75	650	1.00
0.71	0.977	0.13	390	1.00
0.91	0.999	1.38×10^{-2}	89	1.00
0.99	1.000	1.82×10^{-4}	0.76	1.00
0.999	1.000	9.15×10^{-7}	0.23	1.00
1.000	1.000	1.55×10^{-7}	0.10	1.00

1.3 Mass–luminosity relation

Having found the correct value of a number like $(R^3\rho_C/M)$, it must apply for any star, so we deduce how the size and density of a star will change if we alter its mass: $\rho_C \propto M/R^3$. Similarly, from hydrostatic equilibrium, we get $P_C \propto M\rho_C/R$, so that

$$P_C \propto M^2/R^4, \tag{10}$$

which is the relationship we derived earlier in the course when considering the central density of the sun.

The perfect gas law says $P_C \propto \rho_C T_C$, so the four equations of stellar structure give four constant combinations involving the five quantities M, L, R, ρ_C, T_C:

$$\boxed{\frac{\rho_C R^3}{M}; \quad \frac{\rho_C R^4 T_C}{M^2}; \quad \frac{\rho_C M T_C^\alpha}{L}; \quad \frac{\rho_C^\beta M L}{T_C^{4+\gamma} R^4}} \tag{11}$$

These can be manipulated to find out how any given quantity depends on mass. After a little (certainly non-examinable) effort, the luminosity-mass relation comes out as

$$\boxed{L \propto M^{\frac{(3+\alpha)(3+\gamma-\beta)+(2+\alpha)(3\beta-\gamma)}{(3+\alpha)-\gamma+3\beta}}.} \tag{12}$$

This doesn't look very simple, but notice that for Thompson opacity ($\beta = \gamma = 0$), this gives $L \propto M^3$, independent of α (as we had earlier). For large α, we get approximately $L \propto M^{3+2\beta}$, independent of the temperature dependence of the opacity (so $L \propto M^5$ in the large-α Kramer's case).

These simple predictions for the scaling of properties of stars with their mass along the main sequence can be contrasted with the actual properties, as summarised below in Table 5. The overall behaviour is quite close to what we have predicted theoretically: between types M5 and O5, the effective power-law scaling of the stellar properties is $L \propto M^{3.2}$; $R \propto M^{0.67}$; $T_{\rm eff} \propto M^{0.44}$.

Table 5 Physical properties of main-sequence stars

Spectral type	$T_{\rm eff}/{\rm K}$	$M/M_\odot$	$R/R_\odot$	$\bar\rho/\bar\rho_\odot$	$\log_{10}(L/L_\odot)$	M_V
O5	38,000	60.0	12.0	0.035	5.90	−5.7
B0	30,000	17.5	7.4	0.043	4.72	−4.0
B5	16,400	5.9	3.9	0.099	2.92	−1.2
A0	10,800	2.9	2.4	0.21	1.73	+0.6
A5	8,620	2.0	1.7	0.41	1.15	+1.9
F0	7,240	1.6	1.5	0.47	0.81	+2.7
F5	6,540	1.3	1.3	0.59	0.51	+3.5
G0	5,920	1.05	1.1	0.79	0.18	+4.4
G5	5,610	0.92	0.92	1.18	−0.10	+5.1
K0	5,240	0.79	0.85	1.29	−0.38	+5.9
K5	4,410	0.67	0.72	1.79	−0.82	+7.4
M0	3,920	0.51	0.60	2.36	−1.11	+8.8
M5	3,120	0.21	0.27	10.70	−1.96	+12.3

This robust prediction of a steep power law dependence of luminosity on mass is the key theoretical fact underlying the existence of the main sequence. It also governs evolution off the main sequence. Stars can shine only for a time τ in which they convert a significant quantity of their mass into radiation. The total energy radiated is $L\tau$, and this must be of order the nuclear efficiency ($\sim 10^{-3}$) of Mc^2:

$$\boxed{\tau \propto M/L \propto M^{-(2+2\beta)}.} \tag{13}$$

In other words, although massive stars have more fuel, they use it up more quickly, and are the first to evolve away from the main sequence. This lifetime is about 9 Gyr for the Sun, which is therefore about half-way through its lifespan.

2 Convective instability

We now need to come clean about a complication that has been neglected so far: energy is not always transmitted in stars by diffusion of photons. This is because there is a large radial temperature gradient in stars, and we can immediately see a potential problem: hot fluid rises. In other words, there is a possibility of **convection** in which the central parts of stars 'boil' and send streamers of hot material outwards, thus transporting energy far more efficiently than diffusion.

Consider a small parcel of fluid in the star, of mass ΔM; this starts in equilibrium, so that the pressure forces acting on it balance gravity, according to the equation of hydrostatic equilibrium. Now suppose that this parcel is displaced upwards (to larger r). In order to stay in hydrostatic equilibrium at its new radius, $r + dr$ it would need to have the same density as the rest of the fluid at that radius: $\rho(r + dr)$. If it is denser, it will sink, but if it is lighter it will float to larger r and the motion will be unstable. Which of these happens depends on the the magnitude of dT/dr. Let the parcel move a distance Δr, so that the surrounding density is now

$$\rho = \rho_0 + (d\rho/dr)\Delta r, \tag{14}$$

where ρ_0 is the starting density of star and parcel. Note that $d\rho/dr$ is negative. If the change in density of the parcel is larger in magnitude than this, convection will happen:

$$|\Delta\rho_{\rm parcel}| > |(d\rho/dr)\Delta r| \quad \Rightarrow \quad \text{convection} \tag{15}$$

(we use the modulus to avoid worries about signs).

The change in density of the parcel is governed by the fact that the fluid inside suffers **adiabatic** change: there is no time to exchange heat with the surroundings. Therefore, the density and pressure of the parcel are connected by the relation

$$P \propto \rho^\gamma \quad \Rightarrow \quad \Delta \ln \rho_{\rm parcel} = \frac{1}{\gamma}\, \Delta \ln P, \tag{16}$$

where γ is the ratio of specific heats: $\gamma = 5/3$ for a monatomic gas.

The surrounding gas is perfect, so $P = nkT \propto \rho T$. This means that the general gradients in P, T and ρ are related:

$$\frac{d\ln\rho}{dr} = \frac{d\ln P}{dr} - \frac{d\ln T}{dr}. \tag{17}$$

Therefore, since we want $\Delta \ln \rho_{\rm parcel} < (d\ln\rho/dr)\Delta r$ (both changes being negative), the criterion for convection is

$$\frac{1}{\gamma}\,\Delta \ln P < \left(\frac{d\ln P}{dr} - \frac{d\ln T}{dr}\right)\Delta r, \tag{18}$$

and taking the limit of infinitesimal Δr gives

$$\boxed{\left|\frac{d\ln T}{dr}\right| > \left(1 - \frac{1}{\gamma}\right)\left|\frac{d\ln P}{dr}\right| \quad \Rightarrow \quad \text{convection.}} \tag{19}$$

The question is thus whether there could be one or more **convective zones** in side the star, and this depends on the detailed solutions of the structure for diffusive energy transport. In practice, convection is important in the cores of stars above a few $M_\odot$: this is because CNO burning is important in these hotter stars. CNO is so temperature sensitive that one can think of the energy generation as occurring almost in a central delta function, and the central temperature gradient becomes very high. Most stars also become convective in their outer parts: for the Sun, convection sets in at a radius of about $0.7R_\odot$. This complication of convection does not greatly alter the general picture of the scaling of stellar properties with mass, but it certainly makes detailed study of stars a much messier problem.

3 The Virial Theorem

We have already discussed the approximate equality between the kinetic and gravitational potential energies in stars. Now let us look at this *virial theorem* in more detail.

The equation of hydrostatic equilibrium states that $dP/dr = -GM(r)\rho(r)/r^2$. Consider multiplying both sides of this equation by $4\pi r^3$, and integrating the equation from $r=0$ to $r=R$:

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = -\int_0^R \frac{GM(r)\rho(r)4\pi r^2}{r} dr \tag{20}$$

Considering the right–hand side of this equation, since $4\pi r^2\rho(r)dr$ is simply the mass of gas, dm, between r and $r+dr$, then

$$-\int_0^R \frac{GM(r)\rho(r)4\pi r^2}{r} dr = \int_{m=0}^{m=M} \frac{GM(r)}{r} dm \tag{21}$$

and so this is simply the total gravitational potential energy of the system: $E_{\rm grav}$.

The left–hand side of the equation can be integrated by parts:

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = \left[P(r)4\pi r^3\right]_0^R \; - \; 3\int_0^R P(r)4\pi r^2 dr \tag{22}$$

The first of these terms is zero since $P(r)=0$ at $r=R$. The second term is just three times the integral of pressure over the entire volume of the star, and hence can be written as $-3\langle P\rangle V$, where $\langle P\rangle$ is the volume–averaged pressure. Hence, Equation 20 reduces to

$$\langle P\rangle = -\frac{1}{3}\frac{E_{\rm grav}}{V} \tag{23}$$

which says that the average pressure of a system in hydrostatic equilibrium is one-third of the density of gravitational energy. This is the virial theorem.

We now need to relate the macroscopic properties of the gas with its microscopic properties, in order to relate the average pressure of a system with the thermal (kinetic) energy of the particles. To do this, consider a particle in a box of volume $V=L^3$, and let it have velocity $v=(v_x,v_y,v_z)$ and momentum $p=(p_x,p_y,p_z)$. As the particle bounces around the box, it will strike one of the walls perpendicular to the x–axis with a frequency $v_x/2L$, and each such strike will result in the particle's momentum changing by $2p_x$. The force ($\equiv$ rate of change of momentum) on the wall is therefore p_xv_x/L. From N such particles in the box, the total pressure ($\equiv$ force /area) will be

$$P = \frac{1}{L^3}\sum_N p_xv_x \quad = \frac{N}{L^3}\langle p_xv_x\rangle \tag{24}$$

where the brackets indicate mean values. N/L^3 is the particle density, n, and for an isotropic gas, $\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle = \langle pv \rangle /3$. Therefore,

$$P = \frac{n}{3} \langle pv \rangle . \tag{25}$$

For **non–relativistic particles**, $p = mv$, so

$$P = \frac{n}{3} \left\langle mv^2 \right\rangle \quad = \frac{2}{3} \frac{E_{\text{kinetic}}}{V}. \tag{26}$$

The pressure is equal to two-thirds of the kinetic energy density. Substituting this result into Equation 23, and considering also that the total energy (E_{tot}) is simply the sum of the kinetic and the gravitational potential energies, the following relations result:

$$2E_{\text{kinetic}} + E_{\text{grav}} = 0 \tag{27}$$

$$E_{\text{tot}} = -E_{\text{kinetic}} \tag{28}$$

$$E_{\text{tot}} = \frac{1}{2} E_{\text{grav}} \tag{29}$$

These equations, for an ideal gas of non-relativistic particles in hydrostatic equilibrium under their own gravity, are of fundamental importance in astrophysics. A number of implications follow directly from them. Here we will consider just two. First, a system in hydrostatic equilibrium is *bound*, that is it has negative total energy. Also, the more tightly bound it is, the higher the internal kinetic energy has to be: hence as a gas cloud collapses to form a star, it becomes hot. Second, changes in kinetic and gravitational energy are related: this is why the opacity, which controls the rate at which energy escapes from the stellar surface, also therefore controls the rate of fusion in a star. In equilibrium, the rate of energy production in the core balances the rate of radiation energy losses from the surface. Were the fusion rate to increase, then more energy would be generated than lost, and the total energy of the star would increase. This would result in an increase in the gravitational energy, and a decrease in the kinetic energy: ie. the star would expand and cool. These changes would then decrease the fusion rate, hence returning the star to equilibrium.

Note that for **relativistic particles**, the situation is somewhat different because the relation between kinetic energy and momentum is $E = pc$, and $v = c$, so the pressure relates to the kinetic energy as:

$$P = \frac{n}{3} \langle pc \rangle \; = \frac{1}{3} \frac{E_{\text{kinetic}}}{V}. \tag{30}$$

For such particles it then follows that

$$E_{\text{kinetic}} = -E_{\text{grav}} \tag{31}$$

and hence that

$$E_{\text{tot}} = 0 \tag{32}$$

Hydrostatic equilibrium for relativistic particles is therefore only possible if the binding energy is zero. This is an inherently unstable situation, and any object under these conditions is liable to be disrupted. For main sequence stars the gas is non-relativistic, so this is not relevant for gas pressure, but the total pressure of a star does not depend only on the gas pressure, but also on the *radiation pressure*. Radiation pressure is not too important in low-mass stars including the Sun, but as we'll see below, it becomes increasingly important in higher mass stars, particularly in their outer layers. Photons are obviously relativistic, and when radiation pressure dominates the total pressure, stars begin to become unstable. It is this which sets the upper limit to the mass that a star can have on the main sequence, as we shall now derive.

4 The upper and lower mass limits to the Main Sequence

4.1 The Eddington Limit, and the upper mass limit of stars

A given photon, of energy E, has a momentum E/c. Therefore, the rate at which a scattering particle with cross-section σ acquires momentum is σ/c times energy flux density:

$$\dot{p} = \frac{\sigma L}{4\pi r^2 c}. \tag{33}$$

This rate of momentum transfer is a force, and it cannot be allowed to exceed the gravitational force, otherwise the surface layers of the star will be blown away:

$$\sigma L/(4\pi r^2 c) < GMm/r^2. \tag{34}$$

Since both these depend on $1/r^2$, there is a maximum luminosity for a given mass, which is the **Eddington luminosity**:

$$L_{\mathrm{Edd}} = \frac{4\pi GMmc}{\sigma} \tag{35}$$

If the opacity is formed mainly by electron scattering, then the appropriate cross-section is σ_T and we get the maximum rate of radiation

$$\boxed{L_{\mathrm{Edd}} = \frac{4\pi GMm_pc}{\sigma_T} = 10^{4.5}\left(\frac{M}{M_\odot}\right)\ L_\odot} \tag{36}$$

(the particle mass is m_p rather than m_e because the protons supply effectively all the inertia, even though the electrons do all the scattering). As a side note: humans violate this limit by many powers of 10. The reason we aren't blown apart is that the analysis neglects other forces, particularly the interatomic bonds that hold us together.

Since stars at the upper end of the main sequence satisfy

$$L/L_\odot \simeq \left(\frac{M}{M_\odot}\right)^3, \tag{37}$$

we see that stars cannot exceed about $100\,M_\odot$ without blowing themselves apart. This is very much an upper limit, since ionic opacity adds to Thomson scattering. In practice, O stars with $M \gtrsim 30\,M_\odot$ lose a large fraction of their mass through the radiation-driven winds in the course of their main-sequence lifetimes.

4.2 The onset of fusion, and the lower mass limit to the main sequence

To consider the lower mass limit to the main sequence, we need to consider the processes that lead to the formation of a star. Stars start life as gas clouds, which collapse under gravity. At first this collapse will occur freely, unopposed, on the *free–fall timescale* ($t_{FF} = (3\pi/32G\rho)^{1/2}$), with the gravitational energy released being either freely radiated away or used to dissociated hydrogen molecules into atomic hydrogen, or to ionise the atomic hydrogen. Once a significant amount of hydrogen is ionised, however, the gas cloud will become opaque to radiation and the gravitational energy released during collapse will remain within the cloud, raising the kinetic energy. The cloud will approach hydrostatic equilibrium, and collapse will then proceed on the *Kelvin–Helmholtz* timescale, which is the timescale over which a body can radiate away its binding energy:

$$L \sim \frac{\partial}{\partial t}\left(\frac{GM^2}{R}\right) \quad \Rightarrow \quad t_K \left(\equiv \frac{R}{|\dot{R}|}\right) \sim \frac{GM^2}{RL}. \tag{38}$$

We derived this timescale earlier for the Sun to be 3×10^7 years, showing the need for a non-gravitational source of energy.

Once the star is in hydrostatic equilibrium then, as discussed earlier, its gravitational potential energy and its kinetic energy are related via the Virial theorem: $E_{\rm grav} + 2E_{\rm kinetic} = 0$. As the star collapses, its gravitational potential energy goes down and its kinetic energy goes up, leading to a rise in the temperature. For main sequence stars like the sun, this process will continue until the temperature has become high enough that nuclear burning begins. However, this can be prevented from happening if **electron degeneracy** sets in first and halts the collapse. We shall discuss electron degeneracy in detail later on, but will consider a simple approach here.

The uncertainty principle imposes a limit on the range of position that a particle with a given range of momentum may occupy. In 1D,

$$(\Delta x)(\Delta p_x) \gtrsim \hbar. \tag{39}$$

Since at most two electrons, for the two spin states, can occupy a given quantum state in space, there is therefore a limit to how tightly packed electrons of a given range of momenta may be made. The minimum volume that an electron can be squeezed in to is given by roughly

$$(\Delta x)^3 \simeq (\hbar/p_x)^3. \tag{40}$$

Equipartition says that $p_x^2/2m_e = kT/2$, so $p_x^2 = m_e kT$, giving

$$n_e = \frac{1}{(\Delta x)^3} \simeq \frac{(m_e kT)^{3/2}}{\hbar^3}. \tag{41}$$

As this density is reached, electron degeneracy sets in, and prevents the gas cloud from collapsing further. Thus, its temperature rises no further, and if the temperature reached is not high enough for fusion reactions to occur then the gas will not achieve stardom.

We can use the virial theorem to investigate when electron degeneracy is important. Expressing the kinetic energy in terms of the number of particles N and the mean temperature T (using equipartition of energy), the virial theorem can be written as

$$2 \times \frac{3}{2}NkT = \frac{\alpha GM^2}{R} \tag{42}$$

where α is a constant of order unity: $\alpha = 3/5$ for a uniform-density sphere.

This can be re–arranged to give

$$kT = \frac{\alpha G M^2}{3NkR} = \frac{G M m_p}{5R} \tag{43}$$

where we have assumed that the star is mainly hydrogen, so that $N = M/m_p$.

Considering Equation 41, and expressing n_e as $n_e = M/(\frac{4}{3}\pi R^3 m_p)$, the degeneracy limit corresponds to

$$\frac{3M}{4\pi R^3 m_p} \simeq \frac{(m_e kT)^{3/2}}{\hbar^3}. \tag{44}$$

Substituting in for R from Equation 43 then gives

$$\frac{3M}{4\pi m_p}\left(\frac{5kT}{GMm_p}\right)^3 \simeq \frac{(m_e kT)^{3/2}}{\hbar^3} \tag{45}$$

or

$$kT \simeq \left(\frac{4\pi}{3}\right)^{2/3} \frac{G^2 m_p^{8/3} m_e}{25\,\hbar^2} M^{4/3}. \tag{46}$$

For a given mass, we can therefore express in practical units an estimate of the maximum temperature that can be attained before degeneracy becomes important:

$$\boxed{T_{\max} \simeq 10^{8.5} \left(\frac{M}{M_\odot}\right)^{4/3} \text{ K.}} \tag{47}$$

Since fusion reaction rates drop exponentially for $T \lesssim 10^6$ K, a minimum mass for normal stars in the range 0.01–0.1 $M_\odot$ might be expected. The accepted figure for this limit when realistic calculations are performed is in fact $0.08 \pm 0.01 M_\odot$. Objects much less massive than this will generate energy only gravitationally, and will therefore cool to a state of virtual invisibility on the Kelvin–Helmholtz timescale. Such objects are known as **brown dwarfs**.

These objects sit on an interesting transition between star and planet, and are potentially very important as they may hide a large proportion of baryonic matter. They are very difficult to find, since it is very hard to measure the masses of faint cool stars directly, but the first brown dwarf was securely identified in 1995, and many more have been identified since.