High Energy Astrophysics, 2011–12

1. Radiation from Accelerating Charges

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1 Revision of Maxwell's Equations

In this first section of the course we start by proving the classical physics result that accelerated charges radiate energy. We derive this beginning with Maxwell's equations. These are:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

$$\nabla \wedge \mathbf{E} = -\dot{\mathbf{B}} \quad (3)$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\dot{\mathbf{E}} \quad (4)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field strengths, ρ is the electric charge density, and $\mathbf{j}$ is the current density. Note that $\dot{} \equiv \partial/\partial t$. Throughout these notes, quantities written in bold are vector quantities, whereas those in normal text are scalars. 4-vectors and 4-tensors will be indicated by subscripts or superscripts.

The first of these four equations is just Gauss's law: that charge is the source of electric field. It follows directly from the definition of divergence, and the equation for the electric field at location $\mathbf{r}$ relative to a charge q: $\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$. The second equation is simply a statement that there are no magnetic monopoles, ie. no sources or sinks of magnetic fields. Lines of magnetic field do not start or finish anywhere, and therefore the divergence is zero. The third equation arises from the Faraday-Lenz law, that a changing magnetic field (e.g. a moving magnet in a coil) produces an electric field. Prior to Maxwell, the fourth equation was Ampère's law, $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$. Maxwell noted that taking the divergence of this implied that $\nabla \cdot \mathbf{j} = 0$ (due to the vector identity $\nabla \cdot (\nabla \wedge \mathbf{B}) = 0$ for any vector field $\mathbf{B}$), which is true in the steady state, but not if electric charges are moving.

If charges are moving then since electric charge must be locally conserved, the *continuity equation* holds:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0} \tag{5}$$

Maxwell fixed the 4th equation to account for this: $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\dot{\mathbf{E}}$ implies that $\nabla \cdot (\nabla \wedge \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{j} + \frac{1}{c^2}\nabla \cdot \dot{\mathbf{E}} = \mu_0 \nabla \cdot \mathbf{j} + \frac{1}{c^2}\dot{\rho}/\epsilon_0 = \mu_0(\nabla \cdot \mathbf{j} + \dot{\rho}) = 0$ as required. This uses the identity $\epsilon_0 \mu_0 = \frac{1}{c^2}$ and Maxwell's first equation.

The zero divergence of $\mathbf{B}$ in Maxwell's second equation implies that a potential can be defined, known as the *Magnetic Vector Potential* ($\mathbf{A}$), such that

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \tag{6}$$

due to the vector identity that $\nabla \cdot (\nabla \wedge \mathbf{A}) = 0$ for any vector field $\mathbf{A}$. Therefore Maxwell's third equation can be written as

$$\nabla \wedge \mathbf{E} = -\nabla \wedge \dot{\mathbf{A}} \tag{7}$$

This implies that

$$\boxed{\mathbf{E} = -\dot{\mathbf{A}} - \nabla \phi} \tag{8}$$

for some scalar field $\phi(\mathbf{x})$. This last term arises as a (sort of) constant of integration, because we can add any term whose curl is zero. Since the curl of the gradient of any scalar field is zero ($\nabla \wedge (\nabla \phi) = 0$ for any scalar field ϕ), any ϕ will do for now. It is the electrostatic potential, which is apparent for a steady-state field with $\dot{\mathbf{A}} = 0$.

1.1 Gauge Invariance

For any given set of physical fields $\mathbf{E}$ and $\mathbf{B}$, the values of $\mathbf{A}$ and ϕ are not unique. This implies that the physics is immune to a certain amount of mixing between $\mathbf{A}$ and ϕ.

To show this, consider changing

$$\mathbf{A} \to \mathbf{A}' \equiv \mathbf{A} + \nabla \psi \tag{9}$$

where ψ is a scalar field. This clearly does not change $\mathbf{B}$, since $\mathbf{B} = \nabla \wedge \mathbf{A}$ and the curl of the gradient of any scalar field is zero. Since $\mathbf{E} = -\dot{\mathbf{A}} - \nabla \phi$ then in order that $\mathbf{E}$ also remains unchanged, there must be a change in ϕ to counteract the change in $\mathbf{A}$:

$$\phi \to \phi' \equiv \phi - \frac{\partial \psi}{\partial t} \tag{10}$$

Then $\mathbf{E} = -\dot{\mathbf{A}} - \nabla \phi$ becomes $\mathbf{E} = -\dot{\mathbf{A}}' - \nabla \phi' = -\left(\dot{\mathbf{A}} + \nabla(\frac{\partial \psi}{\partial t})\right) - \nabla(\phi - \frac{\partial \psi}{\partial t}) = -\dot{\mathbf{A}} - \nabla \phi$, and thus $\mathbf{E}$ is also unchanged.

Changing the potential in this way is known as a **Gauge Transformation**. Note that ψ is a function of position: we are not simply adding a constant global offset to the potentials, or we could just have set $\nabla \psi = 0$.

The Gauge transformation discussed above may look slightly contrived, but it looks a lot more natural if it is expressed in terms of a 4-vector potential,

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right). \tag{11}$$

The Gauge transformation now simply becomes

$$A^\mu \to A'^\mu \equiv A^\mu - \nabla^\mu \psi \tag{12}$$

where ∇^μ is the contravariant 4-vector, $\nabla^\mu = \left(\frac{\partial}{c\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) \equiv \left(\frac{\partial}{c\partial t}, -\nabla\right)$.

1.2 Maxwell's equations in 4-vector form

It is worth pursuing the 4-vector formalism a bit further. The 4-current is

$$J^\mu = (c\rho, \mathbf{j}) \tag{13}$$

(see Tutorial 1 where you will demonstrate this) which gives the continuity equation a very simple form:

$$\nabla_\mu J^\mu = 0 \tag{14}$$

Physical fields arise via the derivative of the 4-potential, which in tensor form is

$$F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu \tag{15}$$

In this format Maxwell's equations then become very elegant:

$$\boxed{\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu} \tag{16}$$

This concise equation encapsulates Maxwell's 1st and 4th equations, with the 2nd and 3rd equations being satisfied automatically by our definition of the 4-potential. It is an exercise for the student to unpack this relation and demonstrate that this is the case. Maxwell's equations written in this way demonstrate an equality of 4-vectors, which are invariant under Lorentz transformations, so Maxwell's equations form a relativistically-correct theory, even though they were formulated before relativity itself.

Note that writing Maxwell's equations in this way makes it clear that they are Gauge invariant under $A^\mu \to A'^\mu \equiv A^\mu - \nabla^\mu \psi$:

$$F'^{\mu\nu} = \nabla^\mu \mathbf{A}'^\nu - \nabla^\nu \mathbf{A}'^\mu = (\nabla^\mu \mathbf{A}^\nu - \nabla^\mu \nabla^\nu \psi) - (\nabla^\nu \mathbf{A}^\mu - \nabla^\nu \nabla^\mu \psi) = F^{\mu\nu} \tag{17}$$

since the $\nabla^2\psi$ terms cancel.

Maxwell's equations can all be derived in a concise top-down manner in this way (see lecture; non-examinable).

2 The wave equations for A and ϕ

Combining Maxwell's first equation ($\nabla \cdot \mathbf{E} = \rho/\epsilon_0$) with Equation 8 ($\mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$) gives:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= -(\nabla \cdot \dot{\mathbf{A}}) - \nabla^2\phi = \frac{\rho}{\epsilon_0} \\
\nabla^2\phi &= -\frac{\rho}{\epsilon_0} - \nabla \cdot \dot{\mathbf{A}} \qquad (18)
\end{aligned}$$

To get a wave equation for ϕ, subtract $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ from both sides:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \qquad (19)$$

This is a wave equation for ϕ, with some source terms on the right-hand side.

A wave equation for $\mathbf{A}$ can be obtained by considering Maxwell's 4th equation, again in conjunction with Equation 8:

$$\nabla \wedge (\nabla \wedge \mathbf{A}) = \mu_0\mathbf{j} + \frac{1}{c^2}\left(-\ddot{\mathbf{A}} - \nabla\dot{\phi}\right) \qquad (20)$$

Using the vector operator identity $\nabla \wedge (\nabla \wedge \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$ gives:

$$-\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0\mathbf{j} - \frac{1}{c^2}\ddot{\mathbf{A}} - \frac{1}{c^2}\nabla\dot{\phi} \qquad (21)$$

which with some re-arrangement gives

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = -\mu_0\mathbf{j} + \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \qquad (22)$$

This again is a wave equation with source terms on the right-hand side. Note the equivalence of the bracketted parts of the source terms in Equations 19 and 22. Because of the Gauge invariance that we discussed earlier, we are free to modify $\mathbf{A} \to \mathbf{A}' \equiv \mathbf{A} + \nabla\psi$ and to choose $\nabla \cdot \mathbf{A}'$ to suit ourselves. Inspection of Equations (19) and (22) shows that they can be simplified considerably by choosing the **Lorentz Condition**

$$\boxed{\nabla \cdot \mathbf{A}' = -\frac{1}{c^2}\frac{\partial\phi'}{\partial t}.} \qquad (23)$$

This is always possible for an appropriate choice of ψ. To demonstrate this, suppose $\mathbf{A}$ and ϕ do *not* satisfy the Lorentz condition. The Gauge transformation gives

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t} = \nabla \cdot \mathbf{A} + \nabla^2\psi + \frac{1}{c^2}\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}, \qquad (24)$$

and by choosing ψ to satisfy

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \qquad (25)$$

we have

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0 \tag{26}$$

as desired. Note that even specifying the Lorentz condition still does not specify $\mathbf{A}$ and ϕ uniquely: we can still add $\nabla\xi$ and $-\partial\xi/\partial t$, provided ξ satisfies the homogeneous (source-free) wave equation $\nabla^2\xi - \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} = 0$.

The advantage of working with the potentials is that the 4 Maxwell equations are reduced to 2 (three if we also impose the Lorentz condition), which we write here to summarise:

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = -\frac{\rho}{\epsilon_0}} \tag{27}$$

$$\boxed{\nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = -\mu_0\mathbf{j}.} \tag{28}$$

Returning to 4-vector notation, the wave equations may be written as

$$\nabla^\nu\nabla_\nu A^\mu = -\mu_0 J^\mu. \tag{29}$$

This can also be seen to arise directly from Equations 15 and 16, noting that in 4-vector form the Lorentz condition is simply

$$\nabla_\mu A^\mu = 0. \tag{30}$$

3 Solutions to the wave equations

We now proceed to solve the wave equation for ϕ. The wave equation for $\mathbf{A}$ can be solved in a similar manner. The ϕ equation is:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{31}$$

The equation is first simplified by Fourier transforming with respect to time. Remember that the Fourier transform and its inverse are:

$$\tilde{\phi}(\mathbf{x},\omega) \equiv \int_{-\infty}^{\infty} \phi(\mathbf{x},t)\exp(i\omega t)dt \tag{32}$$

$$\phi(\mathbf{x},t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi}(\mathbf{x},\omega)\exp(-i\omega t)d\omega. \tag{33}$$

The ϕ wave equation transforms to

$$\nabla^2\tilde{\phi} + \frac{\omega^2}{c^2}\tilde{\phi} = -\frac{\tilde{\rho}}{\epsilon_0} \tag{34}$$

where $\tilde{\rho}(\mathbf{x},\omega)$ is the Fourier transform of ρ. The advantage of this equation is that it is just a differential equation in the spatial dimensions only: no time derivatives remain.

We solve this equation using a *Green Function*. This is the same approach conceptually as working out the potential of a distribution of charges by splitting the charge into point charges, and adding up the potentials from each point charge. The latter (the response to a point charge) is called a Green function.

Consider the solution of the equation for the (Fourier transform of the) potential, at $\mathbf{x}$, of a unit point 'charge' located at $\mathbf{x}'$. We will call this solution $\tilde{G}(\mathbf{x}, \mathbf{x}')$. It satisfies

$$\nabla^2 \tilde{G} + \frac{\omega^2}{c^2} \tilde{G} = -\delta(\mathbf{x} - \mathbf{x}')/\epsilon_0 \tag{35}$$

where δ is a Dirac delta function (see Course summary notes for a recap of the properties of this). If we can solve this equation, then we can solve the general problem, by noting that

$$\tilde{\rho}(\mathbf{x}) = \int \tilde{\rho}(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')d^3\mathbf{x}' \tag{36}$$

The transform of the potential is therefore a sum of the potentials from the point 'charges':

$$\tilde{\phi}(\mathbf{x}) = \int \tilde{\rho}(\mathbf{x}')\tilde{G}(\mathbf{x}, \mathbf{x}')d^3\mathbf{x}' \tag{37}$$

Note that $\tilde{G}$, $\tilde{\rho}$ and $\tilde{\phi}$ are also functions of ω, but we suppress the ω-dependence for clarity.

Let us return to solving for $\tilde{G}(\mathbf{x}, \mathbf{x}')$. The formal proof is rather tedious, but we can work out the solution. Evidently the solution can depend on $\mathbf{x}$ and $\mathbf{x}'$ through $|\mathbf{x} - \mathbf{x}'|$ only, because of the complete spherical symmetry around the charge at $\mathbf{x}'$. So we work in spherical polars (r, θ, φ), in which case derivatives with respect to θ and φ are zero. In spherical polars, the r component of ∇^2 can be expressed in either of two equivalent forms:

$$\nabla^2 \phi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) \tag{38}$$

Using the latter of these gives the following equation for the Green Function:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\tilde{G}\right) + \frac{\omega^2}{c^2}\tilde{G} = -\frac{\delta(\mathbf{r})}{\epsilon_0} \tag{39}$$

Except at $\mathbf{r} = 0$, the right-hand side of this equation is zero, and, multiplying by r, we have a harmonic equation for $r\tilde{G}$,

$$\frac{\partial^2}{\partial r^2}\left(r\tilde{G}\right) + k^2\left(r\tilde{G}\right) = 0 \qquad (r \neq 0) \tag{40}$$

where $k \equiv \omega/c$. This has the solution

$$r\tilde{G} = a\exp(ikr) + b\exp(-ikr) \tag{41}$$

for some a and b (which may be functions of ω).

We set $b = 0$, since this part of the expression represents a spherical wave incoming from infinity (remember the $\exp(-i\omega t)$ dependence implicit in the Fourier transform).

To determine the value of a, we note that in the limit that $kr \to 0$, $r\tilde{G} \to a$, or $\tilde{G} = a/r$. In these circumstances we know that the equation must simplify to Poisson's equation (transformed), whose solution we know: $\tilde{G} = 1/(4\pi\epsilon_0 r)$ (since for a stationary charge q, $\phi = q/4\pi\epsilon_0 r$). Hence, $a = 1/(4\pi\epsilon_0)$ and our solution for a point unit charge is

$$r\tilde{G} = \frac{exp(ikr)}{4\pi\epsilon_0}. \tag{42}$$

For a more general distribution of charges,

$$\tilde{\phi}(\mathbf{x}, \omega) = \frac{1}{4\pi\epsilon_0} \int \tilde{\rho}(\mathbf{x}', \omega) \frac{\exp(i\omega|\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \tag{43}$$

Implicitly writing $\tilde{\rho}(\mathbf{x}', \omega)$ in terms of a Fourier transform of $\rho(\mathbf{x}', t')$, this becomes

$$\tilde{\phi}(\mathbf{x}, \omega) = \frac{1}{4\pi\epsilon_0} \int \int \rho(\mathbf{x}', t') \exp(i\omega t') dt' \frac{\exp(i\omega|\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \tag{44}$$

In order to get the potential we now reverse the Fourier transform of ϕ (see Equation 33), to give

$$\begin{aligned} \phi(\mathbf{x}, t) &= \frac{1}{2\pi} \frac{1}{4\pi\epsilon_0} \int \int \int \rho(\mathbf{x}', t') \exp(i\omega t') dt' \exp(-i\omega t) \frac{\exp(i\omega|\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' d\omega \\ &= \frac{1}{2\pi} \frac{1}{4\pi\epsilon_0} \int \int \int \rho(\mathbf{x}', t') \frac{\exp\left[iw(t' - t + |\mathbf{x} - \mathbf{x}'|/c)\right]}{|\mathbf{x} - \mathbf{x}'|} dt' d^3\mathbf{x}' d\omega. \end{aligned} \tag{45}$$

This can be simplified by integrating over ω first, using one of the relations of the Dirac delta (see Course Summary handout):

$$\int \exp\left[iw(t' - t + |\mathbf{x} - \mathbf{x}'|/c)\right] d\omega = 2\pi\delta(t' - t + |\mathbf{x} - \mathbf{x}'|/c). \tag{46}$$

This allows the t' integral to be done, giving finally

$$\boxed{\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'.} \tag{47}$$

By similar means, we can solve for $\mathbf{A}$:

$$\boxed{\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'.} \tag{48}$$

These are called *Retarded Potentials*, since they depend not on the distribution of charges now, but some time in the past (delayed by the time required for a light signal to propagate from the source to the point of interest to arrive at time t).

3.1 The Distant Zone

Evaluating the retarded potentials is a difficult calculation for the general case, but simplifies a lot in the case where the source is 'distant'. By 'distant', we mean here that the separation of the source and the observer is sufficiently large that we can treat the separation $R \equiv |\mathbf{x} - \mathbf{x}'|$ as approximately a constant, R_0 (see Figure 1).

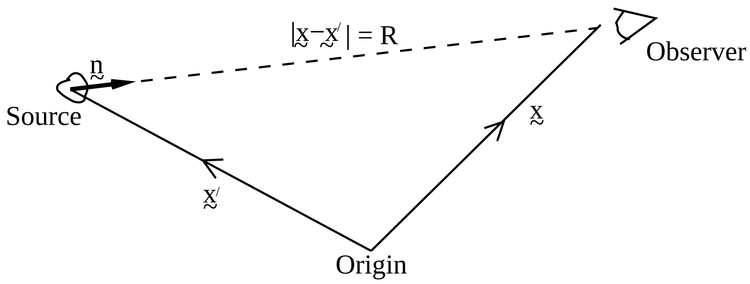

Figure 1: The setup in the distant zone. The emission source is at location $\mathbf{x}'$, and detection is at $\mathbf{x}$. The vector $\mathbf{n}$ is a unit vector pointing from the source to the observer. In the distant zone, the distance $R \equiv |\mathbf{x} - \mathbf{x}'|$ is approximately constant.

This means that small changes in the position of the source ($\mathbf{x}'$) are unimportant in the denominator of the equations for the potential (Equations 47 and 48), and the distance in the denominator can therefore be taken out of the integral. For the ϕ equation this leaves

$$\begin{aligned} \phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0 R_0} \int \rho(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c) d^3\mathbf{x}' && (49) \\ &= \frac{1}{4\pi\epsilon_0 R_0} \int \rho(\mathbf{x}', t - (\mathbf{x}.\mathbf{n} - \mathbf{x}'.\mathbf{n})/c) d^3\mathbf{x}'. && (50) \end{aligned}$$

It is clear that the solution of this will be $\phi(\mathbf{x}, t) = \frac{1}{R_0}$ multiplied by some function of $t - \mathbf{x}.\mathbf{n}/c$:

$$\phi(\mathbf{x}, t) = \frac{1}{R_0} f(t - \mathbf{x}.\mathbf{n}/c) \quad (51)$$

In practice the function f is determined by the precise distribution of sources ρ. Here we will not try to solve this explicitly, but will leave it in this general form. This solution represents a signal propagating radially outwards at speed c.

Similarly, solving for $\mathbf{A}$ would give

$$\mathbf{A} = \frac{1}{R_0} \mathbf{F}(t - \mathbf{x}.\mathbf{n}/c) \quad (52)$$

for some vector function $\mathbf{F}$. This is also a wave propagating outwards at speed c. Since $\mathbf{E}$ and $\mathbf{B}$ are derived from $\mathbf{A}$ and ϕ by differentiation, they are also outgoing waves.

3.2 Energy radiation: E and B from ϕ and A in the distant zone

We want to work out the circumstances in which a charge will radiate energy. The rate of energy radiation is given by the *Poynting Vector* $\mathbf{S} = \mathbf{E} \wedge \mathbf{B}/\mu_0$, so we will need to be able to calculate $\mathbf{E}$ and $\mathbf{B}$ to do this.

$\mathbf{B} = \nabla \wedge \mathbf{A}$, so, in suffix notation,

$$B_i = \epsilon_{ijk} \nabla_j A_k \quad (53)$$

(Suffix notation reminder: indices appearing once on each side of the equation are *free*, and take the values 1,2,3. Repeated indices are *dummy* and are summed over the 3 values. No index may appear more than twice (on each side). ϵ_{ijk} has a value of 1 if i, j, k is a cyclic permutation of 1,2,3 (e.g. 312), a value of -1 if it is an odd permutation (e.g. 132), and it is zero if any 2 indices are the same).

From Equation 52 we can write

$$A_k = \frac{1}{R_0} F_k(W) \tag{54}$$

where we have defined $W = t - \mathbf{x}.\mathbf{n}/c$. Using the chain rule for differentiation $\left(\frac{da}{dx} = \frac{da}{db}\frac{db}{dx}\right)$

$$\nabla_j F_k(W) = F_k'(W)\nabla_j W \tag{55}$$

where $F_k'(W) \equiv dF_k/dW$. Differentiating, $\nabla_j W = \nabla_j(t - x_m n_m/c) = -n_j/c$. Hence

$$B_i = \frac{1}{R_0}\epsilon_{ijk}(-n_j/c)F_k' \tag{56}$$

We can re-order now, since there are no operators any more, so $B_i = -\frac{1}{R_0 c}\epsilon_{ijk} n_j F_k'$. The last part is the i component of $\mathbf{n} \wedge \mathbf{F}'$, so the expression for $\mathbf{B}$ is:

$$\mathbf{B} = -\frac{1}{R_0 c}\mathbf{n} \wedge \mathbf{F}'. \tag{57}$$

To obtain an expression for $\mathbf{E}$ we note that $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$. The $\dot{\mathbf{A}}$ term is simply given by $\dot{\mathbf{A}} = F'(W)/R_0$. For $\nabla\phi$, again using suffix notation, from $\phi = f(W)/R_0$ (Equation 51) we have

$$(\nabla\phi)_i = \nabla_i\left[f(W)/R_0\right] = f'(W)(\nabla_i W)/R_0 = -\frac{n_i}{R_0 c}f'(W). \tag{58}$$

So

$$\nabla\phi = -\frac{1}{R_0 c}f'(t - \mathbf{x}.\mathbf{n}/c)\mathbf{n}. \tag{59}$$

Combining these gives the expression for $\mathbf{E}$:

$$\mathbf{E} = \frac{1}{R_0 c}f'(t - \mathbf{x}.\mathbf{n}/c)\mathbf{n} - \frac{1}{R_0}\mathbf{F}'(t - \mathbf{x}.\mathbf{n}/c). \tag{60}$$

It is worth noting that

$$\mathbf{n} \wedge \mathbf{E} = -\mathbf{n} \wedge \mathbf{F}'/R_0 = c\mathbf{B} \tag{61}$$

(since $\mathbf{n} \wedge \mathbf{n} = 0$), which demonstrates that $\mathbf{B}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{E}$. We can also show that $\mathbf{n}$ and $\mathbf{E}$ are perpendicular to each other, by evaluating $\mathbf{n} \wedge \mathbf{B}$. This gives:

$$\begin{aligned} \mathbf{n} \wedge \mathbf{B} &= -\frac{1}{R_0 c}\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{F}') \\ &= -\frac{1}{R_0 c}\left((\mathbf{n}.\mathbf{F}')\mathbf{n} - \mathbf{F}'\right) \end{aligned} \tag{62}$$

(using the vector identity $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$).

Now, since $\mathbf{A} = \mathbf{F}(W)/R_0$, then $\nabla \cdot \mathbf{A} = -\frac{1}{R_0 c}\mathbf{F}'.\mathbf{n}$. Rearranging this and substituting for $\mathbf{n}.\mathbf{F}'$ into Equation 62 gives

$$\mathbf{n} \wedge \mathbf{B} = -\frac{1}{R_0 c}\left((R_0 c \nabla \cdot \mathbf{A})\mathbf{n} - \mathbf{F}'\right) \tag{63}$$

Noting then the Lorentz condition, $\nabla \cdot \mathbf{A} = -\dot{\phi}/c^2$, and that $\dot{\phi} = f'/R_0$ (from Equation 51), this then becomes

$$\mathbf{n} \wedge \mathbf{B} = -\frac{1}{R_0 c}\left((f'/c)\mathbf{n} - \mathbf{F}'\right) = -\frac{\mathbf{E}}{c}. \tag{64}$$

This demonstrates that $\mathbf{E}$ is perpendicular to $\mathbf{n}$ and hence that $\mathbf{n}$, $\mathbf{E}$ and $\mathbf{B}$ are mutually orthogonal. This can be confirmed by taking scalar products, e.g. $\mathbf{n} \cdot \mathbf{E} = c\mathbf{n} \cdot (\mathbf{n} \wedge \mathbf{B}) = 0$.

Knowing that $\mathbf{n}$, $\mathbf{E}$ and $\mathbf{B}$ are all at right angles makes it easy for us now to calculate the energy flux through the Poynting vector:

$$\begin{aligned} \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B} &= -\frac{c}{\mu_0}(\mathbf{n} \wedge \mathbf{B}) \wedge \mathbf{B} \\ &= \frac{c}{\mu_0}\left[B^2\mathbf{n} - (\mathbf{n} \cdot \mathbf{B})\mathbf{B}\right] \\ &= \frac{c}{\mu_0}\frac{|\mathbf{n} \wedge \mathbf{F}'|^2}{R_0^2 c^2}\mathbf{n} \end{aligned} \tag{65}$$

Since $\mathbf{A} = \mathbf{F}/R_0$ and $\mathbf{F}' = \dot{\mathbf{F}}$, we can write finally

$$\mathbf{S} = \frac{1}{\mu_0 c}|\mathbf{n} \wedge \dot{\mathbf{A}}|^2\mathbf{n} \tag{66}$$

or

$$\boxed{\mathbf{S} = \frac{1}{\mu_0 c}|\dot{\mathbf{A}}_\perp|^2\mathbf{n}} \tag{67}$$

where $\dot{\mathbf{A}}_\perp$ is the component of $\dot{\mathbf{A}}$ perpendicular to $\mathbf{n}$.

4 Radiation from charges in arbitrary motion

So far we have computed the radiated energy from a given magnetic vector potential $\mathbf{A}$. In order to see if charges can radiate energy, we need to compute $\mathbf{A}$ from a charge in arbitrary motion and see what the resulting Poynting flux (ie. energy radiation rate) is.

Consider the world line of a particle as shown in Figure 2. Since the particle travels at speed $v < c$, it crosses the past light cone only once. The only contribution to the integral in the retarded potential equation is from this point in spacetime.

Let $\mathbf{x}_0(t)$ be the path of the charge, whose charge density is therefore $\rho(\mathbf{x}, t) = e\delta(\mathbf{x} - \mathbf{x}_0(t))$. Inserting this into the retarded potential (Equation 47) gives

$$\phi(\mathbf{x}, t) = \frac{e}{4\pi\epsilon_0}\int \frac{\delta(\mathbf{x}' - \mathbf{x}_0(t_{\text{ret}}))}{|\mathbf{x} - \mathbf{x}'|}d^3\mathbf{x}' \tag{68}$$

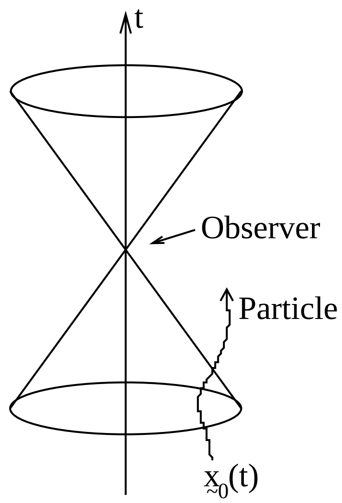

Figure 2: The world-line of a particle crossing the past light cone of an observer.

where $t_{\rm ret} \equiv t - |\mathbf{x} - \mathbf{x}'|/c$ is the retarded time. To simplify this, we introduce an additional integral over time, using a delta function to select only the contribution from the correct retarded time:

$$\phi(\mathbf{x}, t) = \frac{e}{4\pi\epsilon_0} \int\int \frac{\delta(\mathbf{x}' - \mathbf{x}_0(t'))}{|\mathbf{x} - \mathbf{x}'|} \delta(t' - t_{\rm ret}) d^3\mathbf{x}' dt' \tag{69}$$

Now we integrate over the spatial dimensions. We note that since $t_{\rm ret} \equiv t - |\mathbf{x} - \mathbf{x}'|/c$, and hence depends on $\mathbf{x}'$, it is affected by the spatial delta function. The integral gives

$$\phi(\mathbf{x}, t) = \frac{e}{4\pi\epsilon_0} \int \frac{\delta(t' - [t - |\mathbf{x} - \mathbf{x}_0(t')|/c])}{|\mathbf{x} - \mathbf{x}_0(t')|} dt', \tag{70}$$

so we get (as expected) a contribution from only one point in time. To solve this equation, note that the properties of the delta function mean that

$$\int f(y)\delta\left[g(y)\right] dy = \int f\left[y(g)\right]\delta(g)\frac{dy}{dg} dg = \frac{f\left[y(g=0)\right]}{(dg/dy)_{g=0}}. \tag{71}$$

In our case, y is t', $g(y)$ is $t' - t_{\rm ret}$ (ie. $t' = t_{\rm ret}$ at $g = 0$), and $f(y)$ is $1/|\mathbf{x} - \mathbf{x}_0(t')|$, so

$$\begin{aligned}
\phi(\mathbf{x}, t) &= \frac{e}{4\pi\epsilon_0} \frac{1}{(|\mathbf{x} - \mathbf{x}_0(t')|)_{t'=t_{\rm ret}}} \frac{1}{\frac{d}{dt'}\left[t' - t + |\mathbf{x} - \mathbf{x}_0(t')|/c\right]_{t'=t_{\rm ret}}} \\
&= \frac{e}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}_0(t_{\rm ret})|} \frac{1}{\frac{d}{dt'}\left[t' - t + \frac{\mathbf{x}.\mathbf{n}}{c} - \frac{\mathbf{x}_0(t').\mathbf{n}}{c}\right]_{t'=t_{\rm ret}}} \\
&= \frac{e}{4\pi\epsilon_0 R(t_{\rm ret})} \frac{1}{(1 - \beta \cdot \mathbf{n})_{t_{\rm ret}}}
\end{aligned} \tag{72}$$

where $R(t_{\rm ret}) = |\mathbf{x} - \mathbf{x}_0(t_{\rm ret})|$ is the distance of the particle at the retarded time (when it crosses the light-cone), and $\beta \equiv \mathbf{v}/c$.

Similarly, it can be shown that

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0 e}{4\pi R(t_{\rm ret})} \frac{\mathbf{v}}{(1 - \beta \cdot \mathbf{n})_{t_{\rm ret}}} \tag{73}$$

4.1 Total radiated power - Larmor's formula

Now that we have obtained an expression for $\mathbf{A}$, we can use this to obtain the Poynting flux. For non-relativistic motion, we can ignore the β in the denominator of (73), which simplifies things greatly to

$$\mathbf{A} \simeq \frac{\mu_0 e \mathbf{v}}{4\pi R} \tag{74}$$

so in the charge's instantaneous rest frame, the Poynting flux is

$$\mathbf{S} = \frac{1}{\mu_0 c}|\dot{\mathbf{A}}_\perp|^2\mathbf{n} = \frac{\mu_0 e^2}{(4\pi R)^2 c}|\dot{\mathbf{v}}_\perp|^2\mathbf{n} = \frac{\mu_0 e^2 |\dot{\mathbf{v}}|^2 \sin^2\theta}{(4\pi R)^2 c}\mathbf{n}. \tag{75}$$

where θ is the angle between $\dot{\mathbf{v}}$ and $\mathbf{n}$.

This equations shows that *accelerated charges radiate energy.*

Integrating over the full sphere at radius R (surface element is $2\pi R^2 \sin\theta d\theta$), we get **Larmor's (non-relativistic) formula** for the radiated power:

$$\boxed{P = \frac{e^2|\dot{\mathbf{v}}|^2}{6\pi\epsilon_0 c^3}} \tag{76}$$

Note that this is a non-relativistic formula - the charge has to be moving slowly. In the next section, we will modify the formula so that it applies to particles travelling at any speed. We will need this, as we will find that synchrotron radiation comes from electrons travelling very close to the speed of light.

4.2 Relativistic effects

To solve for the total radiated power of a relativistic charge, we could return to Equation 73 and calculate the Poynting flux incorporating the β factors. However, there is a much more elegant method to solve this, using a 4-vector approach.

In the instantaneous rest frame of the charge, the Larmor formula (Equation 76) may be written

$$dE = -\frac{e^2}{6\pi\epsilon_0}\frac{\dot{v}^2}{c^3}dt. \tag{77}$$

We seek a relativistically-correct equation which reduces to Equation 77 in the non-relativistic limit. i.e. we want an equation which equates 4-vectors. We note that (apart from factors of c) dE and dt are both zero-components of 4-vectors, (the 4-momentum $dP^\mu = (dE/c, d\mathbf{p})$ and the prototype 4-vector $dx^\mu = (cdt, d\mathbf{x})$). We therefore look for an equation of the form

$$dP^\mu = -\kappa dx^\mu \tag{78}$$

where κ is a scalar (ie. an invariant quantity). This has the right form for the zero-components, but we need to check the components 1,2,3. In the instantaneous rest frame of the charge, $d\mathbf{x} = \mathbf{0}$, and also the radiation emitted is symmetric (dipole pattern), which has no net momentum, so $d\mathbf{p} = \mathbf{0}$. Hence, in the non-relativistic limit, everything is OK.

The scalar κ must reduce, in the rest frame, to $\frac{e^2}{6\pi\epsilon_0}\frac{\dot{v}^2}{c^5}$ (the extra 2 powers of c coming from the factors in $dx^0 = cdt$ and $dP^0 = dE/c$). Since this involves the square of the acceleration, it is sensible to consider to consider the 4-acceleration, a^μ, and in particular whether the invariant quantity $a^\nu a_\nu$ might fit the bill. The 4-acceleration is $a^\mu \equiv d^2x^\mu/d\tau^2 = dU^\mu/d\tau$, where $U^\mu = \gamma(c, \mathbf{v})$ is the 4-velocity, and τ is proper time. Hence

$$a^\mu = \gamma \frac{d}{dt}\gamma(c, \mathbf{v}) = \gamma(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\dot{\mathbf{v}}). \tag{79}$$

In the instantaneous rest frame, $\gamma = 1$ and $\dot{\gamma} = \gamma^3 \mathbf{v} \cdot \dot{\mathbf{v}}/c^2 = 0$ (since $\mathbf{v} = 0$), so $a_\mu = (0, \dot{\mathbf{v}})$ in the rest frame. Hence the invariant quantity $a^\nu a_\nu$, which can be evaluated in any frame, has the value $-\dot{\mathbf{v}}^2_{\text{rest frame}}$.

Therefore the relativistic generalisation of Larmor's formula is

$$dP^\mu = \frac{e^2}{6\pi\epsilon_0 c^5} a^\nu a_\nu dx^\mu \tag{80}$$

and, in particular, the zero component gives the **Relativistic Larmor Formula** for the radiated power:

$$\boxed{\frac{dE}{dt} = \frac{e^2}{6\pi\epsilon_0 c^3} a^\nu a_\nu.} \tag{81}$$

We can apply this equation classically to any process which accelerates charges, to calculate the rate at which energy is radiated away.

4.3 Enhanced energy loss rates for relativistic particles

Having derived this equation, it may be convenient to 'unpack' the 4-acceleration and write it in terms of the more familiar 3-acceleration $\mathbf{a} = \dot{\mathbf{v}}$. From Equation 79, we have

$$a^\nu a_\nu = \gamma(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\mathbf{a})\gamma(c\frac{d\gamma}{dt}, -\mathbf{v}\frac{d\gamma}{dt} - \gamma\mathbf{a}). \tag{82}$$

Also, $d\gamma/dt = \gamma^3 \mathbf{v} \cdot \mathbf{a}/c^2$. Some algebra (exercise for the student) gives

$$a^\nu a_\nu = -\gamma^4 \left[\mathbf{a}^2 + \gamma^2 \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c}\right)^2\right] \tag{83}$$

so

$$\frac{dE}{dt} = -\frac{e^2\gamma^4}{6\pi\epsilon_0 c^3} \left[\mathbf{a}^2 + \gamma^2 \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c}\right)^2\right]. \tag{84}$$

It may be useful to split $\mathbf{a}$ into components parallel and perpendicular to $\mathbf{v}$, in which case

$$\boxed{\frac{dE}{dt} = -\frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left(a_\perp^2 + \gamma^2 a_\parallel^2\right).} \tag{85}$$

Comparing this with the non-relativistic Larmor equation, we see that the energy loss rate is enhanced for relativistic particles by a high power of γ. Furthermore, the energy loss is greater, for given 3-acceleration, if the acceleration is parallel to $\mathbf{v}$.

4.4 Searchlight beaming of radiation

At additional effect for the radiation from relativistic particles is that of beaming of the radiation. Radiation is a dipole in the electron instantaneous rest frame (S'), and by symmetry half of the radiation is emitted into $|\theta'| < \pi/2$ ($\cos\theta' > 0$) regardless of the angle between the velocity and the acceleration (see Figure 3).

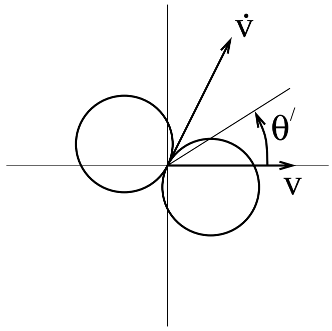

Figure 3: The symmetric dipole radiation pattern of a charge in its instantaneous rest frame.

In the observer's frame (S), however, the dipole radiation is distorted by relativistic aberrations. The relation between θ and θ' (angles in frames S and S') is given by:

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'} \tag{86}$$

This equation is most easily derived from the Lorentz transformation of the photon 4-momentum; see Tutorial Sheet 1.

In this observer's frame, half of the radiation is emitted within an angle θ to the direction of motion, such that $\cos\theta > v/c$. For v close to c, θ is small, and we can expand

$$\cos\theta \simeq 1 - \frac{\theta^2}{2} > \frac{v}{c} = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} \simeq 1 - \frac{1}{2\gamma^2} \tag{87}$$

i.e.

$$|\theta| \lesssim \frac{1}{\gamma} \tag{88}$$

So the radiation is *beamed* forward, within a very small angle to the electron's motion, if γ is large.

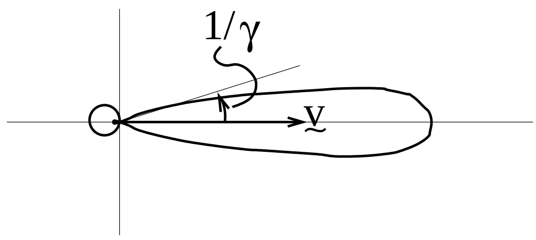

Figure 4: The beamed dipole radiation pattern of a charge moving at relativistic speed. Note that the beamed dipole pattern is not completely symmetric about the x-axis (unless the acceleration in the rest frame is perpendicular to the velocity), due to the shape of the dipole pattern in the rest-frame (see Figure 3).