PHYSICS OF STARS & NEBULAE

TUTORIAL 3

PROBLEM 1: IDEAL GAS – DEGENERATE GAS BOUNDARY

The equation of state for an ideal gas is

$$P_{\rm ideal} = \frac{\rho k_{\rm B} T}{\bar{m}},$$

where ρ is the total mass density of the gas, $k_{\rm B} = 1.38 \times 10^{-23}$ JK^{-1} is the Boltzmann constant, and $\bar{m} = 0.5 m_{\rm H}$ is the mean mass per particle, where $m_{\rm H} = 1.67 \times 10^{-27}$ kg.

The equation of state for a fully degenerate non-relativistic gas is

$$P_{\rm deg} = K_{\rm nr} \rho^{5/3},$$

where

$$K_{\rm nr} = \frac{h^2}{20 m_{\rm e}^{8/3}} \left(\frac{3}{\pi}\right)^{2/3} \left(\frac{\rho_{\rm e}}{\rho}\right)^{5/3},$$

where $\rho_{\rm e}$ is the mass density of electrons, $m_{\rm e} = 9.11 \times 10^{-31}$ kg is the mass of an electron, and $h = 6.63 \times 10^{-34}$ Js is Planck's constant.

The equation of state for a fully degenerate relativistic gas is

$$P_{\rm deg} = K_{\rm r} \rho^{4/3},$$

where

$$K_{\rm r} = \frac{hc}{8 m_{\rm e}^{4/3}} \left(\frac{3}{\pi}\right)^{1/3} \left(\frac{\rho_{\rm e}}{\rho}\right)^{4/3},$$

where $c = 2.998 \times 10^{8}$ ms^{-1} is the speed of light.

(a) Evaluate $K_{\rm nr}$. By comparing the equation of state for an ideal gas with that of a degenerate non-relativistic gas, derive the minimum density ρ required for a gas of temperature T to be degenerate.

(b) Evaluate $K_{\rm r}$. By comparing the equation of state for an ideal gas with that of a degenerate relativistic gas, derive the minimum density ρ required for a gas of temperature T to be relativistically degenerate.

(c) Show that if a gas is already non-relativistically degenerate, that by compressing it to a sufficiently high critical density, the gas will become relativistically degenerate. Show that this critical density is independent of the gas temperature and derive its value.

PTO

PROBLEM 2: DEGENERATE STARS

(a) Brown dwarfs are stars that have masses and central temperatures just below the requirements of hydrogen thermonuclear fusion. They are visible shortly after they form, but as they cool they grow dim until they become invisible. It is possible that brown dwarfs comprise a large fraction of the mass in the Galaxy.

Consider a brown dwarf of mass $M = 0.01 M_\odot$ and radius $R = 0.1 R_\odot$, where $M_\odot = 2 \times 10^{30}$ kg is a solar mass and $R_\odot = 7 \times 10^{8}$ m is a solar radius. Estimate the internal temperature of the brown dwarf by equating the thermal energy per unit mass ($\frac{3}{2} k_{\rm B} T / \bar{m}$) to the gravitational energy per unit mass of the star. Is the interior of the brown dwarf (non-relativistically) degenerate? (For a polytrope satisfying $P \propto \rho^{5/3}$, the central density is 6 times the average density of the star.)

(b) Consider a neutron star of mass $M = 1.4 M_\odot$ and radius $R = 10$ km. Estimate its internal temperature by equating the thermal energy per unit mass to the gravitational energy per unit mass of the neutron star. Is the interior of the neutron star relativistically degenerate? (For a polytrope satisfying $P \propto \rho^{4/3}$, the central density is 54 times the average density of the star.)