PHYSICS OF STARS & NEBULAE

TUTORIAL 2: NUCLEAR FUSION

The rate R_{AB} at which nuclei of atomic numbers Z_A and Z_B will fuse by quantum tunnelling depends on temperature according to

$$R_{AB} \propto T^{-3/2} \int_0^\infty S(E)p(E)dE,$$

where $S(E)$ is a slowly varying function of the kinetic energy of the nuclei, and $p(E)$ describes the probability for fusion, balancing the Boltzmann distribution of particle energies against the probability for tunnelling. (Nuclei with higher kinetic energies fuse more easily, but according to the Boltzmann factor, there are fewer of them.) The coefficient $T^{-3/2}$ comes from the Maxwellian distribution of particle speeds, and accounts for the relative speed of the colliding nuclei (reaction rate is $\propto n\,\sigma\,v$), where σ is the fusion cross-section. The principal temperature behaviour of $p(E)$ is given by

$$p(E) = \exp\left[-\frac{E}{k_{\rm B}T} - \left(\frac{E_G}{E}\right)^{1/2}\right],$$

where E_G is the Gamow energy $E_G = (\pi\alpha Z_A Z_B)^2 2m_{\rm red}c^2$, $\alpha \simeq 1/137$ is the fine–structure constant, and $m_{\rm red} = m_A m_B/(m_A + m_B)$ is the reduced mass of the nuclei.

(a) Estimate the dependence of R_{AB} on temperature T by approximating $p(E)$ as a Gaussian (or Normal) distribution of the form

$$p(E) \propto \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right],$$

where E_0 is the energy E at which $p(E)$ is a maximum, and Δ is the effective width of $p(E)$. This may be done by defining $g(E) = \ln p(E)$, and finding the energy E_0 that maximizes $g(E)$. (It will be the same as the energy that maximizes $p(E)$ – why?) Expand $g(E)$ about this maximum energy E_0 to second order using a Taylor series to work out the width Δ.

(b) Since $p(E)$ is sharply peaked about $E = E_0$ with a width Δ, a good approximation for the fusion rate is $R_{AB} \propto T^{-3/2}p(E_0)\Delta$. Using this, show that

$$\frac{d\ln R_{AB}}{d\ln T} \simeq -\frac{2}{3} + \left(\frac{E_G}{4k_{\rm B}T}\right)^{1/3}.$$

(c) Show that $E_G = 8 \times 10^{-14}$ J for the pp chain. Use this with the result from part (b) to show that for temperatures T near 2×10^7 K, $R_{AB} \propto T^{3.5}$.

(d) Show that $E_G = 5 \times 10^{-12}$ J for the CNO cycle. Use this with the result from part (b) to show that for temperatures T near 2×10^7 K, $R_{AB} \propto T^{16}$.