# Postgraduate astronomy reading group

# Hand-in problems 2007/08

(1) [for October 11]
A simplified galaxy model consists of a uniform slab of stars mixed with dust. Calculate the surface brightness observed as a function of inclination and show that it is independent of inclination in the limit that the amount of dust becomes very large. Specifically, let the slab have a thickness $L$, and let $\tau$ denote the optical depth for a light ray passing perpendicularly through the slab. Assume the emissivity of starlight, $\epsilon$, and the density of dust to be constant throughout the slab (assume that the dust has no scattering opacity; roughly how might the answer change if scattering was important?).

(2) [for Oct 18]
(a) A star, with an apparent magnitude of $m_V = 2.5$, is found to have a parallax of 0.002 arcsecond. What is its absolute magnitude in $V$? Given that the absolute magnitude of a main-sequence A0 star is typically $M_V = +0.6$, what type of star might have this value of $M_V$?

(b) The star explodes and goes supernova, increasing in luminosity by a factor of 50,000. What are the new values of apparent and absolute magnitude?

(c) The supernova is observed in three filters, and is found to have the colours $B - V = 1.0$ and $B - R = 1.5$. In which filter does the object have the highest flux, assuming that the zeropoints for $B$, $V$, and $R$ are $6.6 \times 10^{-9}$, $3.6 \times 10^{-9}$, and $1.7 \times 10^{-9}$ erg cm$^{-2}$ sec$^{-1}$ Å$^{-1}$ respectively.

(d) The supernova remnant, in the shape of a ring, is expanding with a velocity of 10,000 km/sec. After a day, would it be possible to resolve it with a ground based telescope, and why?

(3) [for Oct 25]
(a) The effect of phase errors on a telescope (either from mirror imperfections or atmospheric turbulence) can be modelled by assuming that the aperture is divided into a number of patches, with a different independent phase error in each. Show that the on-axis intensity compared to its value in the absence of errors is reduced by a factor of the expectation value of $\cos(\Delta\phi)$, where $\Delta\phi$ is the phase difference between two patches. Hence show that the practical criterion for optical perfection (50% of ideal intensity) is an rms phase error of approximately 0.7 radians. How does this relate to the common criterion $\lambda/20$ for the rms smoothness of a mirror?

(b) A radio telescope consists of $10^4$ small dishes arranged in a regular grid within a 1 km square, where all pairs of dishes are correlated. What is the FWHM of the synthesised beam as a function of wavelength, and how much sharper does the beam become if the telescope uses uniform weighting of the $UV$ plane, rather than natural weighting?

(4) [for Nov 8]
(a) Given a set of independent measurements $x_i$ of some quantity $x$, with errors $\sigma_i$, show that the optimal way to average the data is with weights $w_i \propto 1/\sigma_i^2$.

(b) An astronomical image $d(i,j)$ is assumed to consist of some true pixel values for a source $f(i,j)$, plus white noise at each pixel $n(i,j)$. You are allowed to weight each pixel value by an arbitrary amount $w(i,j)$ to yield a 'signal' $S = \sum_{i,j} w(i,j)\, d(i,j)$ for the presence of a source. Show that the optimal weight is $w(i,j) \propto f(i,j)$ – i.e. that object detection should proceed by convolving the image with the expected profile of a single object (hint: consider the expectation of $S$ and its variance. Don't work in Fourier space).

(c) You are given several images of the same piece of sky: each has the same noise, but different seeing. Show that the optimal way to combine these images is to convolve each by its PSF and then add them (hint: think in Fourier space, and use the result of part [a]).

(5) [for Nov 15]
(a) Starting with either the density of states in 6-dimensional phase space, derive the equation of state for a fully degenerate highly relativistic pure electron gas:

$$p = K\rho^{4/3},$$

where $p$ is pressure, $\rho$ is the total mass density, and $K$ is a constant. Express $K$ in terms of fundamental physical constants, and show how the result changes if neutrons are substituted for electrons. How does $p$ scale with $\rho$ is the gas is non-relativistic?

(b) By comparing the degenerate equation of state with the ideal gas law, show that the degree of degeneracy of a partially degenerate non-relativistic neutron gas increases as a function of $\rho/T^{3/2}$, where $T$ is the density and temperature of the gas.

(6) [for Nov 22]
Consider an axially symmetric magnetic field that changes slowly in the $z$ direction. Since the field has zero divergence, deduce that the radial component at radius $r$, $B_r(r)$ is

$$B_r = -\frac{r^2}{2}\frac{\partial B_z}{\partial z}.$$

(7) [for Nov 29]
Consider a uniform spherical cloud of ionised hydrogen of mass $M$ , collapsing isothermally at temperature $T$, with radius $R(t)$. It starts optically thin, but becomes optically thick at some radius.

1. What is the Bremsstrahlung luminosity when the cloud is optically thin? (take the Gaunt factor to be $g = 1.2$).
2. What is the luminosity after it is optically thick?
3. By equating these, find the radius at which the transition occurs.

(8) [for Dec 6]
(a) Making the simplifying assumption that a synchrotron-emitting electron emits at the single frequency given by

$$\nu = \gamma^2 \nu_g$$

(where $\nu_g$ is the non-relativistic gyro-frequency), show that the effective temperature of electrons emitting synchrotron radiation at frequency $\nu$ is given by

$$T_e = \left[\tfrac{m_e c^2}{3k}\left(\tfrac{\nu}{\nu_g}\right)^{0.5}\right].$$

(b) Compare this frequency dependence with the frequency dependence of brightness temperature (assume $S_\nu \propto \nu^{-\alpha}$) and explain clearly why a synchrotron source must become self-absorbed below a certain frequency.

(c) Prove that, in the self-absorbed region of the spectrum,

$$S_\nu \propto \nu^{5/2}$$

and explain why this differs from the $\nu^2$ dependence observed in the low-frequency radio spectra observed from compact HII regions.

(9) [for Jan 17]

(a) The Jeans length is given (approximately) by sound speed times free fall collapse time. Explain the meaning and significance of this remark.

(b) If a uniform spherical cloud – not initially pressure-supported – is allowed to collapse without energy loss, what will happen? Give an expression for the temperature, and the relationship between kinetic and potential energy, at the end of this phase.

(c) Suppose at this stage the cloud starts to freely radiate, and maintains constant temperature, what happens and how does the ratio of Jeans length to radius behave? What may be the consequence?

(d) In an interstellar cloud in this last situation, what is likely to halt the process? (hint: how is dust optical depth determined?)

(e) If sound speed is fixed (constant temperature) at $100\,\mathrm{m\,s^{-1}}$ and the dust absorption cross section is equivalent to $10^{-25}\,\mathrm{m}^2$ per H atom (in a cloud composed predominantly of Hydrogen), show that the fragment mass resulting from the above sequence of events is of order 0.1 $M_\odot$.

(10) [for Jan 31]
The equations of stellar structure are:

$$\text{(1) Equation of Continuity : } \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\text{(2) Equation of Hydrostatic Equilibrium : } \quad \frac{1}{\rho}\frac{dP}{dr} = -\frac{GM(r)}{r^2}$$

$$\text{(3) Equation of Energy Generation : } \quad \frac{dL}{dr} = 4\pi r^2 \epsilon \rho$$

$$\text{(4) Equation of Radiative Diffusion : } \quad \frac{L}{4\pi r^2} = -\frac{4ac}{3\rho\kappa} T^3 \frac{dT}{dr}$$

(a) Give the form of the equation of radiative diffusion in the case where the opacity is dominated by electron scattering. For what class of stars is this a good approximation?

(b) Assuming the perfect gas law, show that the scaling of the structure equations in this case imply the following relations between the total mass, total luminosity, radius, central density and central temperature: $\rho_c R^3 \propto M$ (from continuity); $\rho_c R^4 T_c \propto M^2$ (from hydrostatic equilibrium); $ML \propto T_c^4 R^4$ (from radiative diffusion). Hence show that stars dominated by electron scattering should obey $L \propto M^3$ independent of the mechanism of energy generation.

(11) [for Feb 21]
(a) Models of AGN accretion disks suggest that much of the X-ray radiation is liberated at a radius of $R \simeq 5R_s$, where $R_s$ is the Schwarzschild radius. Ignoring relativistic effects and viscosity, show that if an object free-falls from infinity (and all its kinetic energy is released as radiation at $5R_s$) it will liberate $\sim 10\%$ of its rest mass energy.

(b) An AGN has a 0.5–2 keV X-ray luminosity of $10^{39}$ Watts. Use the Eddington limit to derive a lower limit to the mass of the central black hole. Assume a spectrum of $f_\nu \propto \nu^{-1}$ from the far-IR (100 microns) to hard X-rays (100 keV).

(c) Assuming the results of (a) and (b), what is the minimum time it would take a seed black hole (10 solar masses) in the centre of a proto-galaxy to grow by accretion at the Eddington rate to a size sufficient to power this quasar.

(12) [for Feb 28]
(a) Write down Friedmann's equation for the scale factor of the universe, $R(t)$. Assuming that the universe contains pressureless matter and vacuum energy only, rewrite the equation in terms of the density parameters $\Omega_m$ and $\Omega_v$.

(b) Hence show that, if $\Omega_m = 0.1$ and $\Omega_v = 1.5$, then there could not have been a big bang. What is the highest redshift we could expect to see? (This will require some numerical experimentation with your calculator).

(c) The equation for a radial null geodesic in the Robertson-Walker metric is $dr = c\,dt/R(t)$. Using the relation between redshift and scale factor, $1 + z \propto 1/R(t)$, plus Friedmann's equation, deduce the differential relation between comoving distance and redshift:

$$R_0\,dr = \frac{c}{H_0}\,\left[(1-\Omega)(1+z)^2 + \Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2}\,dz.$$

(d) Integrate this expression for the case of the $\Omega = 1$ Einstein–de Sitter universe. In this model, calculate the apparent angle subtended by a galaxy of proper diameter 30 kpc, as a function of redshift (recall $c/H_0 = 3000\,h^{-1}$Mpc). Show that there is a critical redshift at which this angle has a minimum value.