Cosmological Physics: additional topics

1.1 Inflationary matching

The full history of the scale factor in an inflationary universe is worth considering in detail. Start by assuming that the energy density is a mixture of radiation and vacuum energy only, and furthermore consider only a flat model with $k = 0$. This last step can be justified from the Friedmann equation:

$$\dot{R}^2 - \frac{8\pi G}{3}\,\rho\, R^2 = -kc^2. \tag{1.1}$$

If the model expands sufficiently far, the vacuum density will dominate and $\rho\, R^2$ will increase with time – having a minimum value at roughly the point when the densities in radiation and vacuum are equal. If curvature is unimportant at this point, it can always be neglected. We can therefore write the Friedmann equation in terms of a scale factor normalized so that $a(t)$ is unity at some critical time t_1:

$$\dot{a}^2 = \frac{8\pi G}{3}\left(\rho_1^r a^{-4} + \rho_1^v\right) a^2. \tag{1.2}$$

An obvious choice for t_1 is the crossover point where $\rho_1^r = \rho_1^v$, and an obvious unit for time is $\tau \equiv (8\pi G \rho_1^v/3)^{-1/2}$. With these units, and using dashes to denote $d/d(t/\tau)$, we get

$$a'^2 = a^{-2} + a^2 \quad \Rightarrow \quad (a^2)'/2 = \sqrt{1 + (a^2)^2}, \tag{1.3}$$

which is simply integrated to yield

$$\boxed{a = \sqrt{\sinh(2t/\tau)}.} \tag{1.4}$$

This expression neatly interpolates between $a \propto t^{1/2}$ in the early radiation era, and $a \propto \exp t/\tau$ in a late-time de Sitter phase. The radiation-vacuum crossover between these regimes happens at $t_1/\tau \simeq 0.441$.

As is well-known, the de Sitter phase can solve the horizon problem by stretching a small causally-connected patch to a size large enough to cover the whole presently-observable universe, provided it continues for long enough. If the characteristic energy scale of inflation is at the GUT level, then at least 60 e-foldings are required. To illustrate what happens next, suppose that the vacuum energy drops abruptly to zero. In a realistic model, this would of course take some time to happen, but the event can be brief enough that we can model it as a discontinuity. Assume that this event takes place at a time $t/\tau = T$: we have to match a and $\dot{a}$ at the join (otherwise the acceleration form of Friedmann's equation would be singular). Before T, we have $a = \sqrt{\sinh(2t/\tau)}$; after T, we have just radiation-dominated growth – but we should not assume that this is just $a \propto t^{1/2}$. In order to match the boundary conditions, a shift in the origin of time is needed:

$$a = A(t/\tau - B)^{1/2}, \tag{1.5}$$

where it is easy to show that $A = \sqrt{2\cosh 2T}$ and $B = T - (1/2)\tanh 2T$. This is the radiation-dominated solution that applies after the vacuum has decayed. If we make observations after this point, the model has the appearance of a standard hot big bang, with an origin of time at $t/\tau = B$. This solution is plotted in figure 1.1, which illustrates a common error. As we have

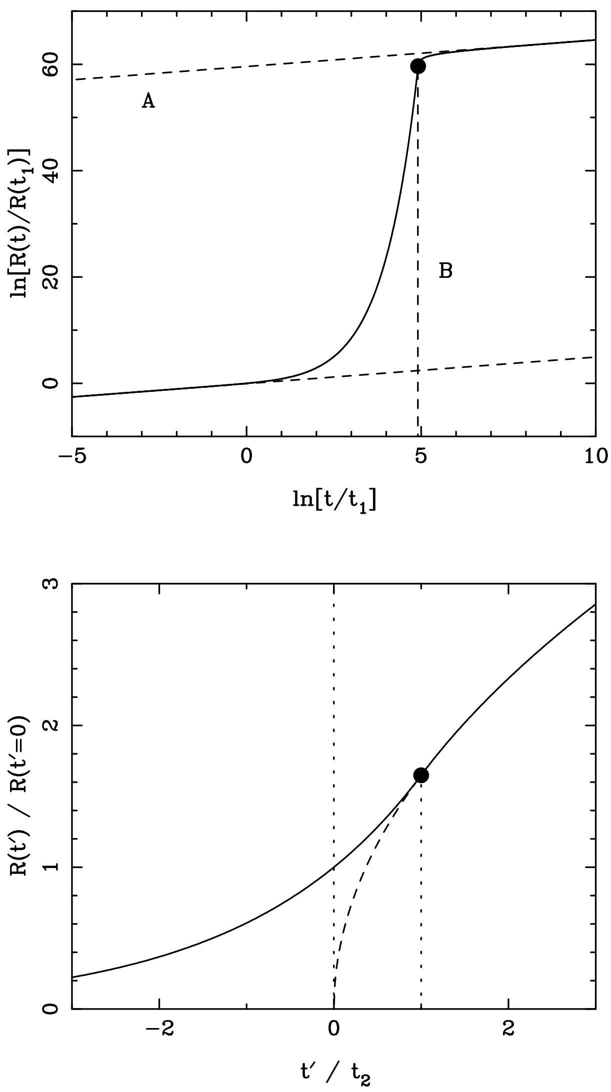

Figure 1.1. The true history of the scale factor in a simple inflationary model. In the top panel, the universe starts out with a singularity at $t = 0$; it contains radiation and vacuum energy, and these reach equal density at time t_1. The subsequent near-exponential behaviour is then assumed to increase the scale factor by a factor exp(60), following which the equation of state abruptly changes to zero vacuum density. This must occur in such a way as to match R and $\dot{R}$, leading to the solid curve, where the plotted point indicates the join. The post-inflation phase is exactly radiation dominated, but with a shift in the origin of time, so that the line marked 'A' is an incorrect (but commonly encountered) extrapolation of the post-inflation phase back in time. The correct extrapolation if vacuum energy is ignored is line 'B', which emerges from an apparent singularity shortly before the time of vacuum decay. This is shown in detail in the second panel, where the origin of time is explicitly shifted to place the apparent singularity at $t' = 0$. The first radiation phase can be ignored, so the universe stays in an exponential de Sitter phase for an indefinite time until the vacuum decays at time $t' = t_2$. For $0 < t' < t_2$, the dashed curve indicates the time dependence we would infer if vacuum energy was ignored: the classical 'big bang'. The inflationary solution clearly removes this feature, placing any singularity at large negative time. The true model expands much *less* rapidly than the big-bang extrapolation, and is much older. This is one way of seeing how the horizon problem can be evaded.

shown, the post-inflation behaviour is not $a \propto t^{1/2}$, and so it is incorrect to extrapolate the final radiation phase back in time using this law. This would give the line labeled 'A', which lies above the true $a(t)$ curve, suggesting that the inflationary universe is always smaller than we would expect from post-inflation data. The correct extrapolation is the line 'B', which emerges from a singularity just before the time of vacuum decay, and which always lies below the full solution.

We can illustrate this more clearly by zooming in on the part of the time axis around the apparent singularity, which we take to occur at a new time coordinate $t' = 0$. The problem can be simplified by assuming that the inflationary phase is exactly exponential, so we have to match $a \propto \exp Ht'$ to $a \propto (t')^{1/2}$, at a time $t' = t_2$; it is then easy to show that $t_2 = 1/2H$. The solution in this form is shown in the second panel of figure 1.1. When we observe the universe at $t' > t_2$, we predict that there was a singularity at $t' = 0$, but the real universe existed far earlier than this. In principle, the question 'what happened before the big bang?' is now answered: there was no big bang. There might have still been a singularity at large negative t', but one could imagine the de Sitter phase being of indefinite duration, so that the true origin of everything can be pushed back to $t' = -\infty$. This idea of a non-singular origin to the expansion dates back to E.B. Gliner (1966: Sov. Phys. – JETP, 22, 378).

This expanded plot of $a(t)$ presents the history of the universe in a manner that challenges the way inflation is commonly described. Guth's phrase 'an extraordinarily rapid expansion' is correct as stated, but it obscures the fact that the rate of expansion would have been even greater were it not for the vacuum energy. Compared to the $a \propto t^{1/2}$ of a standard hot big bang, the inflationary universe is in fact an extraordinarily *slow* expansion. Indeed, this feature is exactly what is needed in order to solve the horizon problem: the universe is actually much older than we would infer from observations in the radiation era, so establishing causal contact is much less of a problem.