Astrophysical Cosmology 4 2004/2005

Problem set 5

(1) Consider an expanding universe that contains a fluid with a relativistic equation of state: $p = u/3$, where u is the energy density. By considering conservation of energy in a volume $\propto R(t)^3$, show that the mass density scales as $\rho \propto R(t)^{-4}$.

(2) Write down Friedmann's equation for the evolution of the cosmic scale factor, $R(t)$. If the mass density is dominated by a relativistic fluid, derive the relation between cosmological time and density (argue that curvature can always be neglected at early times).

(3) The universe currently contains black-body radiation with $T = 2.73$ K. Calculate the contribution of this radiation to the density parameter (express your result in terms of the dimensionless Hubble parameter, h. You will need the value of the Stefan–Boltzmann constant, which is $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$). Hence deduce the redshift at which the densities of radiation and non-relativistic matter were equal (expressed as a function of Ω_m and h).

(4) The phenomenon of neutrino freezeout means that the universe should also contain three species of neutrinos with a temperature $(4/11)^{1/3}$ smaller than that of the photons. Show that this boosts the total relativistic content by a factor 1.68, and deduce a revised redshift of matter-radiation equality.

(5) Using your previous results, estimate the age of the universe at matter-radiation equality, if $\Omega_m = 0.3$ and $h = 0.7$. Hence estimate the proper size of the 'horizon length' at that time, by evaluating ct. What value does this length take when expressed in comoving coordinates? (i.e. what size does this length expand to today?).

(6) Contrast this approximate calculation of the comoving horizon length at matter-radiation equality with the exact result, derived using the equation for a radial null geodesic in a flat universe:

$$R_0\,dr = \frac{c}{H_0}\,\left[\Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2}\,dz.$$