Astrophysical Cosmology 4 2004/2005

Problem set 2

(1) Write down Friedmann's equation for the scale factor of the universe, $R(t)$. Assuming that the universe contains pressureless matter only, so that $\rho(t) = \rho_0 (R/R_0)^{-3}$, rewrite the equation in terms of the variable η, where $d\eta = c\,dt/R(t)$ and show

$$(R/R_*)'^2 = 2(R/R_*) - k(R/R_*)^2,$$

where $R_* = 4\pi G \rho_0 R_0^3 / 3c^2$.

(2) Hence verify that $R(\eta) = R_*(1 - \cos\eta)$ and $R_*(\cosh\eta - 1)$ are the solutions for respectively positive and negative space curvature. Use $dt = R\,d\eta/c$ to deduce $t(\eta) = (R_*/c)(\eta - \sin\eta)$.

(3) If $H_0 = 70\,\mathrm{km\,s^{-1}Mpc^{-1}}$ and we inhabit a matter-dominated universe with $\Omega = 1.1$, where $R_* = (c/H_0)\,\Omega\,|1-\Omega|^{-3/2}/2$ $R_0 = (c/H_0)\,|1-\Omega|^{-1/2}$, then

(a) What is the current age of the universe?
(solve $R(\eta)$ for η_0 and use $H_0^{-1} = 9.78\,h^{-1}\mathrm{Gyr}$)

(b) How much larger will the universe be when it ceases expanding?

(c) When will this happen?

(d) What will $H(t)$ and $\Omega(t)$ be at this time?

(e) At what time will the big crunch happen?

(4) Suppose instead that we inhabit a universe with a significant vacuum energy. If the matter and vacuum densities are $\Omega_m = 0.1$ and $\Omega_v = 1.5$, show that there could not have been a big bang. What is the highest redshift we could expect to see? (This will require some numerical experimentation with your calculator).

[PTO]

(4) [1997 Summer exam]
The dynamics of the expanding isotropic universe can be summarized in terms of the time evolution of the Scale Factor $R(t)$. Using simple Newtonian dynamics, show that

$$\ddot{R} = -\frac{4\pi G R \rho}{3}$$

$$\dot{R}^2 = \frac{8\pi G \rho R^2}{3} + 2E,$$

where $\rho = \rho(t)$ is the density of the Universe at time t, and E is the total energy (per unit mass) of the Universe.

4 marks

In the Einstein-de Sitter Universe the total energy $E = 0$. Adopting a present-day value for the Hubble Constant of $H_0 = 65\,\mathrm{kms^{-1}Mpc^{-1}}$, calculate the present-day matter density of an Einstein de-Sitter Universe.

4 marks

Next calculate the effective present-day matter density of the microwave background ($T = 2.73$ K), and hence deduce the redshift z_{eq} at which the density of radiation was the same as that of matter.

4 marks

From the energy conservation equation given above, derive the formula for time t as a function of density ρ and redshift z in the case of a radiation-dominated Einstein-de Sitter Universe. **6 marks**

Use this formula to work out the time t_{eq} of radiation-matter equality, and then the time at which nucleosynthesis of the light elements commenced (at a temperature $T = 10^{10}$ K).

6 marks

$G = 6.673 \times 10^{-11}\mathrm{Nm^2kg^{-1}}$
$k = 1.381 \times 10^{-23}\mathrm{JK^{-1}}$
$1\mathrm{pc} = 3.086 \times 10^{16}\mathrm{m}$
$c = 2.998 \times 10^{8}\mathrm{ms^{-1}}$