Astrophysical Cosmology 4   2004/2005

Solution set 7

(1) Solve the problem of matching an inflationary vacuum-dominated de Sitter expansion with $R \propto \exp(Ht)$ onto a radiation-dominated era via a sudden change in the equation of state. R and $\dot{R}$ must match at the join (why?). If this change is assumed to take place at a time t_c after the classical big bang, show that the condition $t_c = 1/2H$ is required, so that the two solutions can be written as

$$\frac{R(t)}{R(t=0)} = \begin{cases} \exp(t/2t_c) & (t < t_c) \\ \sqrt{e\,(t/t_c)} & (t > t_c) \end{cases} .$$

Solution: We need to match R and $\dot{R}$ at t_c in order that there is no jump in $\dot{R}$. If there were we would not have a finite $\ddot{R}$ at the match, whereas we know that $\ddot{R} \propto \rho + 3p$ is finite. At the transition from inflation to a radiation-dominated universe at time t_c we must then have $R(t_c) = R_0 e^{Ht_c} = At_c^{1/2}$ and $\dot{R}(t_c) = R_0 H e^{Ht_c} = (1/2)At_c^{-1/2}$. Taking the ratio of these shows that $H(t_c) = (1/2)t_c^{-1}$. Hence from $R(t_c)$ we find $e^{Ht_c} = e^{1/2} = At_c^{1/2}/R_0$, so that $A = R_0\sqrt{e/t_c}$. Finally we see that if $R(t)/R_0 = \exp(t/2t_c)$ for $t < t_c$, then $R(t)/R_0 = \sqrt{e(t/t_c)}$ for $t > t_c$.

(2) The largest features measured in large-scale structure today are superclusters of size up to $100\,h^{-1}$ Mpc. Calculate the angle that such structures would subtend if placed on the last-scattering surface at $z = 1100$ (give the result as a function of Ω_m, and consider both models with no vacuum energy, and flat vacuum-dominated models). Given that we possess CMB data to a highest angular multipole of $\ell \simeq 1000$, discuss the observability of infant superclusters as a function of Ω_m.

Solution: The distance to the CMB is given by the comoving horizon size today is

$$D_H = R_0 \int_0^{1100} \frac{dr}{dz} dz \approx R_0 \int_0^{\infty} \frac{dr}{dz} dz = \frac{2c}{H_0} \times \Omega_m^{\alpha}$$

where $\alpha = 1$ for an open ($\Omega_v = 0$) model and $\alpha = 0.4$ for a flat ($\Omega_m + \Omega_v = 1$) model (see notes). The prefactor $2c/H_0 \approx 6000h^{-1}$Mpc. The multipole $\ell = 1000$ corresponds to an angular scale of $\theta = 2\pi/\ell \approx 0.006$ radians. A $100h^{-1}$Mpc proto-supercluster at the horizon subtends $(100/6000)\Omega_m^{-\alpha} = (1/60)\Omega_m^{-\alpha}$ radians. Hence proto-clusters can be seen directly in the CMB if $\Omega_m \geq 0.38$ for a flat model and 0.09 for a flat universe. Given we think the universe is flat and $\Omega_m \approx 0.3$, we can expect to see proto-clusters in the CMB in principle.

A spherical perturbation exists in the present-day universe. Its radius is $50\,h^{-1}$ Mpc and its density is 1.1 times the mean value. Suppose we could observe the perturbation to the CMB temperature caused by the progenitor of such a structure located at $z = 1100$. What mechanisms dominate the temperature fluctuation, and what is the approximate magnitude of $\Delta T/T$? (assume that the universe is the matter-dominated $\Omega = 1$ Einstein-de Sitter model).

Solution: For an Einstein-de Sitter universe the growth of linear density perturbations is given by $\delta \propto a$. If its present-day overdensity is $\delta_0 = 0.1$, its density at the CMB, $z = 1100$, is $\delta = 0.1/1100 = 1.1 \times 10^{-5}$. Its potential field is given by $\Delta\Phi = GM/r$, where $M = (4\pi/3)\delta\rho r^3$. The relative density is given by $\delta\rho = 0.1\bar{\rho}$, where $\bar{\rho} = 2.78 \times 10^{11} M_\odot \Omega h^2 \mathrm{Mpc}^{-3}$ is the mean density of the universe. Hence the mass of the object is $M = 10^{16.16} h^{-1} M_\odot$. The potential now is $\Delta\Phi/c^2 = 10^{-4.86}$. As the linear potential does not evolve in an EdS universe, this will be its value at the CMB. Hence we can estimate the magnitude of the two important effect generating the CMB; (1) Sachs-Wolfe effect; $\Delta T/T = (1/3)\Delta\Phi/c^2 \approx 10^{-5.34}$; (2) the adiabatic effect; $\Delta T/T = (1/3)\delta(z_{CMB}) = 10^{-4.52}$. We see that this object will produce both effects with similar magnitude – but opposite sign. SW cools radiation, as energy is lost climbing out of the potential well, while the adiabatic effect heats the radiation up.

(3) [part of 2001 degree exam question] One hypothesis for the origin of dark matter is that it consists of weakly interacting relic particles. Explain the reason why it is normal to distinguish three types ('hot', 'warm', and 'cold') of such particles, and give the approximate mass scales associated with each case.

A relic dark-matter particle imposes a coherence scale on cosmic structure depending on the time at which it becomes non-relativistic. Given that the age of the universe is approximately $t = 1$ s when its temperature is 10^{10} K, calculate this coherence scale in comoving length units as a function of the mass (express your result in comoving Mpc, using mass units of eV). Hence summarize the astronomical reasons why it is considered unlikely that the universe is dominated by hot dark matter.

Solution: Hot Dark Matter (HDM) is a massive particle species that decouples, or freezes out when its interaction rate drops below the Hubble expansion rate ($\langle\sigma v\rangle n < H(z)^{-1}$), when the particles velocity is still relativistic. An example of HDM is a massive neutrino with $m \sim 100$eV. Warm Dark Matter (WDM), which freezes out early, but with 100× more degrees of freedom and is still relativistic, and has mass $M \sim 10k$eV. There are no major contenders for WDM. Cold Dark Matter (CDM) is non-relativistic when it freezes out, and so is more massive. If its physics is neutrino-like, the number density is exponentially suppressed ($n \sim me^{-m/M_{\mathrm{freeze}}}$) and so has mass $m \sim 100$GeV. The current prime candidate for CDM is the lightest supersymmetric particles, the neutralino. **[5 Marks]**

The coherence scale is set by free-streaming of massive particles out of structure. The proper damping scale from free-streaming is $\sim ct$ at a time when $kT = mc^2$. As $T \propto R^{-1}$ and $R \propto t^{1/2}$ in the radiation-dominated regime,

$$t \approx \left(\frac{T}{10^{10}K}\right)^{-2} \text{seconds}$$

given that $T = 10^{10}$K at $t = 1$ second. The freezout temperature, T, for a particle with rest mass m is $T = mc^2/k$. A 1eV particle has energy $E = mc^2 = 1.602 \times 10^{-19}$ in SI units, and $k = 1.38 \times 10^{-23}$, so $T = 10^{4.1}(m/1\text{eV})$K. Now ct is the proper length, so we need to multiply by $(1 + z) = T/2.73$K to get a comoving distance (ie a distance today. So the comoving coherence scale is

$$L = ct\left(\frac{T}{2.73K}\right) = c\left(\frac{T}{10^{10}K}\right)^{-2} \frac{T}{2.73K}.$$

Which gives us

$$L = 10^{5.55}\text{Mpc}\left(\frac{T}{1K}\right)^{-1}.$$

Substituting T for mass m we find the comoving coherence scale in terms of the particle mass, m;

$$L = 28\text{Mpc}\left(\frac{m}{\text{eV}}\right)^{-1}$$

[10 Marks]

Hence if the universe is dominated by HDM where $m \sim 100$eV we find the comoving coherence scale is that of superclusters. But superclusters are still forming today, and we see galaxies that exist at redshifts of $z > 5$. Hence we need a galaxy-size damping scale, giving $m > 1$keV. **[5 Marks]**