Astrophysical Cosmology 4 2004/2005

Solution set 6

(1) Is it a good approximation to treat the matter in the early stages of the big bang as an ideal gas? Test this by comparing the typical kinetic energy to the electrostatic potential energy between a particle and a neighbour at a typical distance. Recall that the number density for relativistic particles in thermal equilibrium is $n \sim g\,(kT/\hbar c)^3$.

Solution: The typical inter-particle spacing is $n^{-1/3}$. The typical potential energy of interaction for a particles with its nearest neighbour is

$$E_{pot} = \frac{e^2}{4\pi\epsilon_0 n^{-1/3}} \sim g^{1/3} \frac{e^2 kT}{4\pi\epsilon_0 \hbar c}.$$

The typical thermal energy is $E_{therm} = kT$. Taking the ratio

$$\frac{E_{pot}}{E_{therm}} \sim g^{1/3}\alpha$$

where $\alpha = e^2/(4\pi\epsilon_0\hbar c) = 1/137$ is the fine-structure constant. At high temperatures $g \to 100$ (i.e. there are about 100 particles in the standard model), so that the electromagnetic potential energy is much smaller than the thermal energy. Hence it is safe to treat matter as an ideal gas in the early stages of the big bang. Note that we would have reached the opposite conclusion if $g \sim 10^6$, so we need to assume we know the right number of particle species in the early universe.

(2) The redshift of last scattering is approximately 1100, but this assumes that intergalactic matter is largely neutral for lower redshifts. Observationally, this is not true at low redshift, where the gas is ionized by ultraviolet light from stars and quasars. Assume that the baryonic material in the universe is re-ionized suddenly at a redshift z_c, and calculate the resulting optical depth due to Thomson scattering. How large does z_c have to be before this reaches unity, so that typical CMB photons would no longer be scattered at $z = 1100$? You may assume the distance-redshift relation

$$R_0 dr = \frac{c}{H_0}\,\left[(1-\Omega_m-\Omega_v)(1+z)^2 + \Omega_v + \Omega_m(1+z)^3\right]^{-1/2}\,dz.$$

Assume that $z_c \gg 1$.

Solution: The optical depth due to Thomson scattering from electrons with a number density n_e is

$$\tau = \int d\ell \, n_e \sigma_T$$

where $d\ell =$ element of proper length, and σ_T is the Thomson cross-section. The proper length is given by $d\ell = R_0 dr/(1+z)$ and the number density scales as $n_e(z) = n_e^0(1+z)^3$. Hence the optical depth is

$$\tau = \sigma_T n_e^0 \int_0^{z_c} R_0 dr (1+z)^2.$$

At large redshift

$$R_0 dr = \frac{c}{H_0 \Omega_m^{1/2}} \frac{dz}{(1+z)^{3/2}},$$

so the integral is dominated by the high-redshift part. Integrating we find

$$\tau \approx \frac{2c\sigma_T n_e^0}{3H_0 \Omega_m^{1/2}} (1+z_c)^{3/2}.$$

We now need to relate n_e^0 to the baryon density, Ω_B. The density of baryons is given by $\rho_B = \mu m_p n_e$, where $\mu = 1.14$ for 25% Helium by mass, and m_p is the mass of the proton. This gives us $\rho_B - 1.88 \times 10^{-26}$ kg m^{-3}. Given that $\sigma_T = 6.65 \times 10^{-29} m^2$, and $c/H_0 = 3000 h^{-1}$Mpc and $1pc = 3,0856 \times 10^{16} m$, we find that

$$\tau \approx 0.04 \Omega_B \Omega_m^{-1/2} h (1+z_c)^{3/2}.$$

For $\Omega_B = 0.04$, $\Omega_m = 0.3$ and $h = 0.7$ we find that the optical depth is unity for $z_c = 60$. Since we can see the primordial fluctuations intrinsic to last scattering at the CMB, we can put an upper limit of 60 to redshift at which stars and quasars could have re-ionising the universe.

(3) The spherical collapse model represents the radius–time relation of a proto-object as

$$r = A(1 - \cos\theta)$$
$$t = B(\theta - \sin\theta).$$

Show that, if $A^3 = GMB^2$, these relations satisfy $\ddot{r} = -GM/r^2$. Expand these relations up to order θ^5 to show that, for small t:

$$r \simeq \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right].$$

Hence show that the linear-theory perturbation to the density is

$$\delta_{\rm lin} = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}.$$

What prediction does this expression make for the density inside the sphere if it is extrapolated to the point of collapse to a singularity?

Solution: We first want to show that $\ddot{r} = -GM/r^2$ if $A^3 = GMB^2$. The time derivative can be written

$$\frac{d}{dt} = \left(\frac{dt}{d\theta}\right)^{-1} \frac{d}{d\theta} = \frac{1}{B(1-\cos\theta)} \frac{d}{d\theta}.$$

Hence

$$\dot{r} = \frac{A\sin\theta}{B(1-\cos\theta)}.$$

Differentiating again we find

$$\ddot{r} = \frac{-A}{[B(1-\cos\theta)]^2]} = \frac{-A^3/B^2}{r^2}$$

which equals $-GM/r^2$ if $GM = A^3/B^2$. We now want to expand r and t to 5^{th} order in θ to find how r evolves with t. Expanding we find

$$r = A(1-\cos\theta) \approx A(\theta^2/2 - \theta^4/24 + O(\theta^6))$$

and

$$t = B(\theta - \sin\theta) \approx B(\theta^3/6 - \theta^5/120 + O(\theta^7)).$$

So $(6t/B) \approx \theta^3(1-\theta^2/20)$. We can invert this by taking the first order solution, $\theta \approx (6t/B)^{1/3}$. We can find the second order solution by inverting the fifth order solution; $\theta = (6t/B)^{1/3}(1+\theta^2/60)$, and replacing the θ on the RHS with the first order solution, giving

$$\theta = \left(\frac{6t}{B}\right)^{1/3} \left(1 + \frac{1}{60}\left(\frac{6t}{B}\right)^{2/3}\right).$$

we can now insert this in the expansion for r;

$$r = \frac{A}{2}\theta^2(1-\theta^2/12) = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3} \left(1 + \frac{1}{30}\left(\frac{6t}{B}\right)^{2/3}\right) \left(1 - \frac{1}{12}\left(\frac{6t}{B}\right)^{2/3}\right).$$

Multiplying the final two brackets and keeping only the leading term we find

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3} \left(1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right).$$

The leading term here is $r \propto t^{2/3}$, which is the expansion rate of an EdS universe. The second term is then the deviation from this due to the perturbation. The density of the perturbation is given by $\rho \propto 1/r^3$. Hence the fractional overdensity is

$$\frac{\delta\rho}{\rho} = -3\frac{\delta r}{r} = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}.$$

So we find that the overdensity, to linear order, grows as $\delta \propto t^{2/3} \propto a$, i.e. at the same rate as the expansion scale of the universe. This may seem like a rather odd coincidence, but for an EdS universe, the only scale available is the scale length. It's also worth noting that in the linear regime, the growth rate of density perturbations is independent of scale. This turns out to be a consequence of an $1/r^2$ force law (although we shall not prove it here). Note also that during linear growth of perturbations, the perturbation is still expanding, only not as fast as the rest of the universe. Finally, when this object finally does collapse to a singularity, $r = 0$ and $\theta = 2\pi$ and $t = 2\pi B$. The linear overdensity at this time will be

$$\delta_{lin} = \frac{3}{20}(12\pi)^{2/3} \approx 1.69.$$