Astrophysical Cosmology 4 2004/2005

Solution set 5

(1) Consider an expanding universe that contains a fluid with a relativistic equation of state: $p = u/3$, where u is the energy density. By considering conservation of energy in a volume $\propto R(t)^3$, show that the mass density scales as $\rho \propto R(t)^{-4}$.

Solution: The first law of thermodynamics (conservation of energy) tells us that $dE = TdS - PdV$. For adiabatic changes $dS = 0$, so $dE = -PdV$. Lets write $E = uV$, where u is the energy-density. Then $d(uV) = -PdV = -u/3dV$ for relativistic fluids. Expanding the LHS; $d(uV) = Vdu + udV = -u/3dV$, so we can write $Vdu = -(4/3)udV$, or $du/u = -(4/3)dV/V$. Since $V \propto R^3$, then $dV/V = 3dR/R$ so that $du/u = -4dR/R$ and hence $u \propto R^{-4}$.

(2) Write down Friedmann's equation for the evolution of the cosmic scale factor, $R(t)$. If the mass density is dominated by a relativistic fluid, derive the relation between cosmological time and density (argue that curvature can always be neglected at early times).

Solution: Friedmann's equation is

$$\dot{R}^2 = \frac{8\pi G \rho R^2}{3} - kc^2.$$

For a relativistic fluid $\rho \propto R^{-4}$, so that $\rho R^2 \propto R^{-2}$. At small R, we can neglect the constant $-kc^2$ term, so

$$\dot{R}^2 = BR^{-2}$$

where $B = 8\pi G \rho R^4/3$ is a constant. Taking the expanding (positive) square root, we find $dR/dt = \sqrt{B}/R$ or $RdR = \sqrt{B}dt$. Integration give us

$$t = R^2/(2\sqrt{B}) = \sqrt{\frac{3}{32\pi G\rho}}.$$

Note that we still find the timescale is of the form $t \propto 1/\sqrt{G\rho}$, which is true of all gravitational systems.

(3) The universe currently contains black-body radiation with $T = 2.73$ K. Calculate the contribution of this radiation to the density parameter (express your result in terms of the dimensionless Hubble parameter, h. You will need the value of the Stefan–Boltzmann constant, which is $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$). Hence deduce the redshift at which the densities of radiation and non-relativistic matter were equal (expressed as a function of Ω_m and h).

Solution: The energy-density of radiation is given by $\rho = 4\sigma T^4/c^3$. Hence $\rho(T = 2.73K) = 4.676 \times 10^{-31}$kg m^{-3}. Comparing this with $\rho = 1.88 \times 10^{-26}\Omega h^2$kg m^{-3}, from the notes, we find $\Omega_r h^2 = 2.49 \times 10^{-5}$. The energy-density in matter and radiation are equal when $1 + z_{eq} = \rho_0^{\mathrm{matter}}/\rho_0^{rad} = \Omega_m h^2/\Omega_r h^2$. This gives $1 + z_{eq} = 40,208\Omega_m h^2$.

(4) The phenomenon of neutrino freezeout means that the universe should also contain three species of neutrinos with a temperature $(4/11)^{1/3}$ smaller than that of the photons. Show that this boosts the total relativistic content by a factor 1.68, and deduce a revised redshift of matter-radiation equality.

Solution: When electrons and positrons annihilate, $e^+ + e^- \rightarrow \gamma$ we conserved entropy, and we know that $s(\text{Fermions}) = (7/8)s(\text{bosons})$ for the same temperature and number of degrees of freedom. Hence the initial entropy in e^+, e^- and γ is $(2 \times 7/8 + 1)s_\gamma = (11/4)s_\gamma$. Given that $s \propto T^3$, this tells us that the temperature in the radiation field is raised by a factor $(11/4)^{1/3}$ compared to that in the neutrinos (see lecture notes). Since $u(\text{fermions}) = (7/8)u(\text{Bosons})$ and we have 3 species of neutrino the energy-density in the neutrinos is

$$\frac{u_\nu}{u_\gamma} = 3 \times \frac{7}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 = \frac{21}{8}\left(\frac{4}{11}\right)^{4/3} \approx 0.68.$$

So the total energy of the radiation is boosted by the neutrino fraction, and hence $1+z_{eq}$ is divided by a factor 1.68, i.e., there is more radiation around so equality happens later. This changes the redshift of equality to $1 + z_{eq} = 23,933\Omega_m h^2$.

(5) Using your previous results, estimate the age of the universe at matter-radiation equality, if $\Omega_m = 0.3$ and $h = 0.7$. Hence estimate the proper size of the 'horizon length' at that time, by evaluating ct. What value does this length take when expressed in comoving coordinates? (i.e. what size does this length expand to today?).

Solution: Using $H_0^2 = 8\pi G\rho_0^{matter}/(3\Omega_m)$ (from the definition of Ω_m), and that at z_{eq} we have $\rho_{rad} = \rho_m = \rho_0^{matter}(1 + z_{eq})^3$, we can re-write the age of the universe in the relativistic regime by

$$t_{eq} = \frac{1}{2H_0}\Omega_m^{-1/2}(1 + z_{eq})^{-3/2}.$$

Plugging in $\Omega_m = 0.3$ and $h = 0.7$, this gives us

$$t_{eq} = 4.375 \times 10^{-6} H_0^{-1} = 61,000\,\text{years}.$$

The approximate proper size of the horizon length at this time is $ct_{eq} = 61,000$ light years, or 18,700 parsecs. This will have expanded by a factor of $1 + z_{eq}$ by today, giving 65.8 Mpc today.

(6) Contrast this approximate calculation of the comoving horizon length at matter-radiation equality with the exact result, derived using the equation for a radial null geodesic in a flat universe:

$$R_0\, dr = \frac{c}{H_0} \left[\Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2} dz.$$

Solution: The problem with the previous approach is that matter is just becoming important at t_{eq}. The proper comoving distance is given by

$$R_0 r = \int_0^z \frac{c dz}{H(z)},$$

where $H(z)^2 = H_0^2[\Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4]$. We ignore curvature, since this is negligible at high redshift. We may also ignore vacuum energy at high redshift, since this is has only become dominant today. Transforming the integral to $a = (1+z)^{-1}$ we find

$$R_0 r = \frac{c}{H_0} \int_0^a \frac{da}{\sqrt{\Omega_m a + \Omega_r}}.$$

Changing variables again to $y = (\Omega_m/\Omega_r)a$ we find

$$R_0 r_{eq} = \frac{c}{H_0} \int_0^1 \frac{dy}{\sqrt{1+y}} \frac{\Omega_r^{1/2}}{\Omega_m}.$$

This can be integrated to give

$$R_0 r_{eq} = 2(\sqrt{2}-1)\frac{c}{H_0}\Omega_m^{-1/2}(1+z_{eq})^{-1/2}.$$

Evaluating for $\Omega_m = 0.3$ and $h = 0.7$ this is gives a comoving distance of $R_0 r_{eq} = 109.3$ Mpc.