Astrophysical Cosmology 4 2004/2005

Solution set 4

(1) The flux density of an object at redshift z, observed at frequency ν_0, is

$$f_\nu(\nu_0) = \frac{L_\nu([1+z]\nu_0)}{(1+z)\,[R_0 S_k(r)]^2}.$$

Consider a source that emits a thermal spectrum of temperature T. If observations are made at low frequencies ($h\nu_0 \ll kT$), show that the flux density *increases* with redshift for $z \gtrsim 1$, until a critical redshift is reached. Give an order-of-magnitude estimate for this redshift in terms of ν_0 and T.

Solution: For a thermal source, the flux is proportional to that of a black-body, ie

$$L_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}.$$

For low frequencies ($h\nu \ll kT$) this is the Rayleigh-Jeans part of the spectrum. Expanding to leading order we find $L_\nu \propto \nu^2$, up to the turnover in the spectrum at $h\nu \sim kT$. Hence in the RJ regime $L_\nu(\nu_0(1+z)) \propto (1+z)^2$. In the denominator we also have $[R_0 S_k(r)]^2$. But as $z \to \infty$, $S_k(r) \to$ constant, since we come up against the horizon distance (recall that this is because we are looking back in time until we hit the initial singularity). Hence we find

$$f_\nu(\nu_0) \propto (1+z)$$

for $z \gg 1$. At low redshift we recover the Euclidean result that $f_\nu(\nu_0) \propto 1/z^2$ for $z \ll 1$. Hence the flux density initially falls with distance, then starts to increase again, as we the spectrum in redshifted into the observed frequency range. This carries on until the spectrum is redshifted past the cut-off at $h\nu_0(1+z) \sim kT$. Hence the flux drops rapidly for thermal objects at a critical redshift of $1 + z_c \sim kT/h\nu_0$. This result is used for the detection of sub-millimeter galaxy sources, which have a thermal spectrum. These can be as easily detected at redshift $z \approx 10$ as at $z = 1$, because of this effect.

(2) The 'deceleration parameter' is defined as a dimensionless form of the second time derivative of the scale factor: $q \equiv -\ddot{R}R/\dot{R}^2$. Use the acceleration form of Friedmann's

equation to obtain an expression for the current value of q, for a universe containing a mixture of vacuum energy and nonrelativistic matter. Show that the expansion decelerates only if $\Omega_m > 2\Omega_v$. Explain qualitatively why objects at a given redshift appear brighter in a decelerating universe and fainter in an accelerating universe.

Solution: Friedmann's equation says

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} - kc^2.$$

while the acceleration equation says

$$\ddot{R} = -\frac{4\pi G R}{3}(\rho + 3P/c^2).$$

Given we can write the Hubble parameter as $H = \dot{R}/R$ the deceleration parameter is given by

$$q \equiv -\frac{\ddot{R}R}{\dot{R}^2} = \frac{4\pi G}{3H^2}(\rho + 3P/c^2).$$

For matter $\rho = \rho_m$ and $P = 0$, while for vacuum energy $\rho = \rho_v$ and $P_v = -\rho_v c^2$. Hence

$$q = \frac{4\pi G}{3H^2}(\rho_m - 2\rho_v)$$

This can be re-written in terms of the density parameters as

$$q = \Omega_m/2 - \Omega_v.$$

The redshift we see depends on the recession velocity of the object. For large q, velocities we larger in the past, giving a higher redshift for a given distance. Hence for a given redshift the distance will be lower, and the object will be brighter.

(3) The equation for a radial null geodesic in the Robertson-Walker metric is $dr = c\,dt/R(t)$, which can be cast in the observational form

$$R_0\,dr = \frac{c}{H_0}\left[(1 - \Omega_m - \Omega_v)(1+z)^2 + \Omega_v + \Omega_m(1+z)^3\right]^{-1/2} dz,$$

for $z \lesssim 1000$. Expand this relation as a series in z to obtain an approximation for the luminosity distance $D_{\mathrm{L}}(z) = (1+z)R_0 S_k(r)$ that is valid to second order in z. Show that the second-order correction depends only on the combination $\Omega_m/2 - \Omega_v$. Hence explain the sense of the near-degeneracy between Ω_v and Ω_m as determined from the supernova Hubble diagram.

Solution: The proper comoving distance is given by

$$R_0 r = \int_0^z \frac{c dz}{H(z)},$$

where $H(z)^2 = H_0^2[(1-\Omega_m-\Omega_v)(1+z)^2+\Omega_v+\Omega_m(1+z)^3]$. The luminosity distance is $D_L(z) = (1+z)R_0S_k(r)$. For small r, $S_k(r) \approx r - kr^3/6$ so equals r to second order. We are asked to calculate $D_L(z)$ to second order in z, which requires second order in $r(z)$. But this only requires us to find $1/H(z)$ in the integrand to first order. Expanding we find

$$H(z) \approx H_0[1+(2+\Omega_m-2\Omega_v)z/2+O(z^2)],$$

so that

$$R_0 r \approx \frac{c}{H_0}(z-(2+\Omega_m-2\Omega_v)z^2/4).$$

Hence the luminosity distance is

$$D_L(z) \approx (1+z)\frac{c}{H_0}(z-(2+\Omega_m-2\Omega_v)z^2/4) \approx \frac{c}{H_0}(z+(2+\Omega_m-2\Omega_v)z^2/4),$$

where our final expression is correct to second order in redshift, and depends on the combination $\Omega_m/2-\Omega_v$. As the supernova Hubble diagram uses the supernova as standard candles, they are really measuring $D_L(z)$ versus z. At low z this gives the relation $D_L(z) = cz/H_0$ ($z \ll 1$), while at higher redshift we deviate from this relation. Hence a fit to the observed supernova data is really a fit to the function $f = 2+\Omega_m-2\Omega_v$, which gives us a set of parallel lines on the Ω_m-Ω_v plane. Hence locus of acceptable values in the Ω_m-Ω_v plane from supernova is degenerate along one of these lines. In practice this degeneracy is not complete, as higher-order terms contain different combinations of Ω_m and Ω_v.

(4) An object is observed at redshift z in a matter-dominated universe with density parameter Ω. Calculate the observed rate of change of redshift for the object (hint: remember $1+z = R_0/R_{\rm emit}$, where both R_0 and $R_{\rm emit}$ change with time, and that time intervals in high-redshift objects are observed to be time-dilated). What fractional precision in observed frequency would be needed to detect cosmological deceleration in a decade?

Solution: For this we need to note that both R_0 and $R_{\rm emit}$ will depend on time, and that we are measuring time in the observers rest frame, $t_{\rm obs}$. Hence

$$\dot{z}_{\rm obs} = \frac{dz}{dt_{\rm obs}} = \frac{dR_{\rm obs}/dt_{\rm obs}}{R_{\rm emit}} - \frac{R_{\rm obs}}{R_{\rm emit}^2}\frac{dR_{\rm emit}}{dt_{\rm obs}}.$$

We can relate the observers time to the emitters time by $dt_{\rm emit} = dt_{\rm obs}/(1+z)$, and using $R_{\rm emit} = R_{\rm obs}/(1+z)$ we find

$$\dot{z}_{\rm obs} = (1+z)H_0 - H_{\rm emit}(z)$$

where $H_0 = \dot{R}_{\rm obs}/R_{\rm obs}$, and $H_{\rm emit} = \dot{R}_{\rm emit}/R_{\rm emit}$. For a matter-dominated universe $H(z) = H_0(1+z)\sqrt{1+\Omega z}$, from the Friedmann equation. Hence we find that

$$\dot{z}_{\rm obs} = -H_0(1+z)(\sqrt{1+\Omega z}-1).$$

This is the observed rate of change of redshift over time for an object at redshift z. The fractional error in accuracy of the measured redshift required to detect this change in a time of $\delta t = 10$ years is

$$\frac{\delta z_{\rm obs}}{(1+z)} = -H_0(\sqrt{1+\Omega z}-1)\delta t.$$

For $z = \Omega = 1$ this is a fractional error in redshift of $-10^{9.4}$. Hence it is far beyond current accuracies to measure the change in redshift.